Super (a,d)-Edge-antimagic Total Labeling of Shackle of Fan Graph

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Abstract

A graph G of order p and size q is called an (a, d)-*edge-antimagic total* if there exist a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that the edge-weights, $w(uv) = f(u) + f(v) + f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study super (a, d)-edge-antimagic total properties of connected of shackle of Fan Graph. The result shows that shackle of Fan Graph admit a super edge antimagic total labeling for $d \in 0, 1, 2$ for $n \geq 1$. It can be concluded that the result of this research has convered all the feasible n, d.

Key Words :*(a, d)-edge-antimagic total labeling, super (a, d)-edge-antimagic total labeling, Fan Graph.*

Introduction

Defnitions of (a,d)-EAT labeling and super (a,d)-EAT labeling were introduced by Simanjuntak at al [7]. These labelings are natural extensions of the notion of edge- magic labeling, dened by Kotzig and Rosa [6], where edge-magic labeling is called magic valuation, and the notion of super edge-magic labeling, is natural extension of the notion of edge-magic labeling dened by Kotzig and Rosa [6]. The super (a, d)-edge-antimagic total labeling [8] is natural extension of the notion of super edge-magic labeling. For more information about graph can be found in [1],[3],[4],[2],[5]. In this paper we will now concentrate on the connected shackle of Fan graph denoted by $\mathbb{F}_n$. The example of figure 1.

Super (a, d)-edge Antimagic Total Labeling

An (a, d)-edge-antimagic total labeling on a graph G is a bijective function f:$V(G) \cup E(G) \to \{1, 2, 3, ...p + q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v)$; $uv \epsilon E(G)$, form an arithmetic progression $\{a, a + b, a + 2b, \ldots, a + (q - 1)d\}$, where $a > 1$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)-edge-antimagic total graph. Such a graph G is called super if the smallest possible labels appear on the vertices.

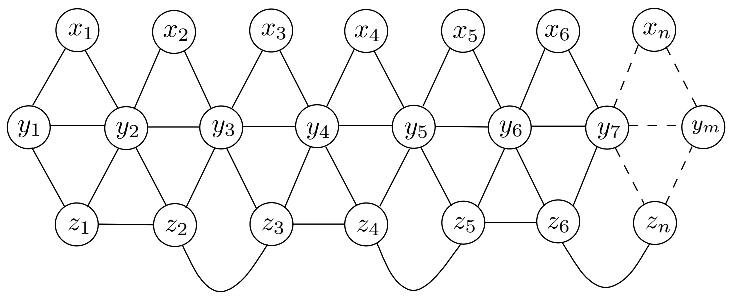

Figure 1:

Thus, a super (a, d)-edge-antimagic total graph is a graph that admits a super (a, d)-edgeantimagic total labeling.

Shackle of fan graph denoted by $\mathbb{F}_n$ with $n \geq 1$ is a connected graph with vertex set. $V(\mathbb{F}\ _n)$=$\{x_i, y_j, z_i; 1 \leq i \leq n; 1 \leq j \leq m; m, n\epsilon N\}$ and $E(\mathbb{F}\ _n)$= $\{x_iy_i, x_iy_{i+1}, y_iz_i, z_iz_{i+1}, z_iy_{i+1}; 1 \leq j \leq n \cup y_jy_{j+1}; 1 \leq j \leq m\}$. Thus $|V(\mathbb{F}\ _n)|$=p=$3n+$ 1 and $|E(\mathbb{F}\ _n)|$=q=$6n - 1$.

We continue this section by a necessary condition for a graph to be super (a, d)-edge antimagic total, providing a least upper bound for feasible values of d.

Lemma 1 *If a (p, q)-graph is super (a, d)-edge-antimagic total then* $d \leq \frac{2p+q-5}{q-1}$

Proof Assume that a (p, q)-graph has a super (a, d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$. The minimum possible edge-weight in the labeling f is at least $1 + 2 + p + 1 = p + 4$. Thus, $a \geq p + 4$. On the other hand, the maximum possible edge-weight is at most $(p - 1) + p + (p + q) = 3p + q - 1$. So we obtain $a + (q - 1)d \leq 3p + q - 1$ which gives the desired upper bound for the difference d. Or we can write:

$$\begin{aligned}
&\Leftrightarrow \quad a + (q-1)d \leq 3p + q - 1 \\
&\Leftrightarrow \quad (p+4) + (q-1)d \leq 3p + q - 1 \\
&\Leftrightarrow \quad d \leq \frac{3p + q - 1 - (p+4)}{q-1} \\
&\Leftrightarrow \quad d \leq \frac{2p + q - 5}{q-1} \\
&\Leftrightarrow \quad d \leq \frac{2(3n+1) + (6n-1) - 5}{(6n-1) - 1} \\
&\Leftrightarrow \quad d \leq \frac{6n + 2 + 6n - 6}{6n-2} \\
&\Leftrightarrow \quad d \leq \frac{12n - 4}{6n-2} \\
&\Leftrightarrow \quad d \leq 2 \\
&\Leftrightarrow \quad d \in \{0, 1, 2\} \qquad (1)
\end{aligned}$$

□

Lemma 2 *A (p,q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p + q + m$, where $m = min(M)$ and $S = \{a - (p+1), a - (p+2), \ldots, a - (p+q)\}$.*

The two above lemma will be used for develop theorem 1.

Result

If shackle of Fan graph has a super (a,d)-edge-antimagic total labeling then for $p = 3n+1$ and $q = 6n-1$ it follows from Lemma 1 that the upper bound of d is $d \leq 2$ or $d \in \{0,1,2\}$. The following Lemma describes an $a,1$-edge-antimagic vertex labeling for shackle of Fan graph.

Lemma 3 *If $n \geq 1$, then the Shackle of Fan graph $\mathbb{F}_n$ has an $(a,1)$-edge-antimagic vertex labeling.*

Proof. Define the vertex labeling $f_1 : V(\mathbb{F}_n) \rightarrow \{1, 2, \ldots, 3n+1\}$ in the following way:

$$\begin{aligned}
f_1(x_i) &= 3i, \text{for } 1 \ \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_1(x_i) &= 3i - 1, \text{for } 1 \ \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
f_1(y_i) &= 3j - 2, \text{for } 1 \ \leq j \leq m \\
f_1(z_i) &= 3i - 1, \text{for } 1 \ \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_1(z_i) &= 3i, \text{for } 1 \ \leq i \leq n \text{ and i } \epsilon \text{ even number}
\end{aligned}$$

The vertex labeling f_1 is a bijective function. The edge-weights of $\mathbb{F}_n$, under the labeling f_1, constitute the following sets

$$
\begin{aligned}
w_{f_1}(x_i y_i) &= 5i-1, \text{for} 1 \leq j \leq n, \\
w_{f_1}(x_i y_{i+1}) &= 6i+1, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
w_{f_1}(x_i y_{i+1}) &= 6i, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
w_{f_1}(y_i z_i) &= 6i-3, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
w_{f_1}(y_i z_i) &= 6i-2, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
w_{f_1}(z_i z_{i+1}) &= 6i+2, \text{for} 1 \leq i \leq n, \\
w_{f_1}(z_i y_{i+1}) &= 6i, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
w_{f_1}(z_i y_{i+1}) &= 6i+1, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
w_{f_1}(y_j y_{j+1}) &= 6j-1, \text{for} 1 \leq j \leq m,
\end{aligned}
$$

It is not difficult to see that the set $w_{f_1} = \{3,\ 4,\ 5, \ldots, 6n-1\}$ consists of consecutive integers. Thus f_1 is a $(3,1)$-edge antimagic vertex labeling.

Bača, Y. Lin, M. Miller and R. Simanjuntak [5], Theorem 5) have proved that if (p,q)-graph G has an (a,d)-edge antimagic vertex labeling then G has a super$(a+p+q, d-1)$-edge antimagic total labeling and a super$(a+p+1, d+1)$-edge antimagic total labeling. With the theorem Lemma 3 in hand, we obtain the following result.

◇ **Teorema 1** *If* $n \geq 1$ *then the graph* $\mathbb{F}_n$ *has a super* $(9n+3, 0)$*-edge-antimagic total labeling and a super* $(3n+5, 2)$*-edge-antimagic total labeling.*

Proof.
Case 1. $d = 0$
We have proved that the vertex labeling f_1 is a $(3,1)$-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling f_1 to a super $(a,0)$-edge-antimagic total labeling, where, for $p = 3n+1$ and $q = 6n-1$, the value $a = 9n+3$.

Case 2. $d = 2$
Label the vertices of $\mathbb{F}_n$ with f_3 that the edge labeling for $d = 2$, so we can that

label the edges with the following way.

$$
\begin{aligned}
f_3(x_iy_i) &= 3n+6i-3, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(x_iy_i) &= 3n+6i-4, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(x_iy_{i+1}) &= 3n+6i, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(x_iy_{i+1}) &= 3n+6i-1, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(y_iz_i) &= 3n+6i-4, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(y_iz_i) &= 3n+6i-3, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(z_iz_{i+1}) &= 3n+6i+1, \text{for} 1 \le i \le n,\\
f_3(z_iy_{i+1}) &= 3n+6i-1, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(z_iy_{i+1}) &= 3n+6i, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(y_jy_{j+1}) &= 3n+6j-2, \text{for} 1 \le j \le m,
\end{aligned}
$$

The total labeling f_3 is a bijective function from $V(\mathbb{F}_n) \cup E(\mathbb{F}_n)$ onto the set $\{1,2,3,\dots,3n+1\}$. The edge-weights of $\mathbb{F}_n$, under the labeling f_3, constitute the sets

$$
\begin{aligned}
W_{f_3} &= \{w_{f_3} + f_3(x_iy_i); \text{ for } 1 \le i \le n\} = 3n+11i-4 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(x_iy_i); \text{ for } 1 \le i \le n\} = 3n+11i-5 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f_3} + f_3(x_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12+1 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(x_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12i-1 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f_3} + f_3(y_iz_i); \text{ for } 1 \le i \le n\} = 3n+12-7 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(y_iz_i); \text{ for } 1 \le i \le n\} = 3n+12i-5 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f3} + f_3(z_iz_{i+11}); \text{ for } 1 \le i \le n\} = 3n+12i+3\\
W_{f_3} &= \{w_{f_3} + f_3(z_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12i-1 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(z_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12i+1 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f3} + f_3(y_jy_{j+1}); \text{ and } 1 \le j \le m\} = 3n+12j-3
\end{aligned}
$$

It is not difficult to see that the set $W_{f_3} = \{3n+5, 3n+7, 3n+9, , \dots, 15n+1\}$ contains an arithmetic sequence with $a = 3n+5$ and $d = 2$. Thus f_3 is a super $(3n+5,2)$-edge-antimagic total labeling. This concludes the proof. □

Theorem 2 *If $n \ge 1$, then the graph $\mathbb{F}_n$ has a super $(6n+4,1)$-edge-antimagic total labeling.*

Proof. Label the vertices of $\mathbb{F}_n$ with $f_4(x_iy_i) = f_1(x_iy_i)$,$f_4(x_iy_{i+1}) = f_1(x_iy_{i+1})$, $f_4(y_iz_i) = f_1(y_iz_i)$,$f_4(z_iz_{i+1}) = f_1(z_iz_{i+1})$, $f_4(y_jy_{j+1}) = f_1(y_jy_{j+1})$, $f_4(z_iy_{i+1}) = f_1(z_iy_{i+1})$ untuk $1 \le i \le n, 1 \le j \le m$ and label the edges with the following way.

$$\begin{aligned}
f_4(x_iy_i) &= 9n-3i+3, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_4(x_iy_i) &= 6n-3i+4, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
f_4(x_iy_{i+1}) &= 6n-3i+5, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_4(x_iy_{i+1}) &= 9n-3i+2, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
f_4(y_iz_i) &= 6n-3i+4, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_4(y_iz_i) &= 9n-3i+3, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
f_4(z_iz_{i+1}) &= 9n-3i+1, \text{for} 1 \leq i \leq n, \\
f_4(z_iy_{i+1}) &= 9n-3i+2, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_4(z_iy_{i+1}) &= 6n-3i+2, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
f_4(y_jy_{j+1}) &= 6n-3j+3, \text{for} 1 \leq j \leq m,
\end{aligned}$$

The total labeling f_4 is a bijective function from $V(\mathbb{F}\ _n) \cup E(\mathbb{F}\ _n)$ onto the set $\{1, 2, 3, \ldots, 3n+1\}$. The edge-weights of $\mathbb{F}\ _n$, under the labeling f_4, constitute the sets

$$\begin{aligned}
W_{f_4} &= \{w_{f_4} + f_4(x_iy_i); \text{ for } 1 \leq i \leq n\} = 9n+2i+2 \text{ and i } \epsilon \text{ odd number} \\
W_{f_4} &= \{w_{f_4} + f_4(x_iy_i); \text{ for } 1 \leq i \leq n\} = 6n+2i+3 \text{ and i } \epsilon \text{ even number} \\
W_{f_4} &= \{w_{f_4} + f_4(x_iy_{i+1}); \text{ for } 1 \leq i \leq n\} = 6n+3i+6 \text{ and i } \epsilon \text{ odd number} \\
W_{f_4} &= \{w_{f_4} + f_4(x_iy_{i+1}); \text{ for } 1 \leq i \leq n\} = 9n+3i+2 \text{ and i } \epsilon \text{ even number} \\
W_{f_4} &= \{w_{f_4} + f_4(y_iz_i); \text{ for } 1 \leq i \leq n\} = 6n+3i+1 \text{ and i } \epsilon \text{ odd number} \\
W_{f_4} &= \{w_{f_4} + f_4(y_iz_i); \text{ for } 1 \leq i \leq n\} = 9n+3i+1 \text{ and i } \epsilon \text{ even number} \\
W_{f_4} &= \{w_{f_4} + f_4(z_iz_{i+1}); \text{ for } 1 \leq i \leq n\} = 9n+3i+3 \\
W_{f_4} &= \{w_{f_4} + f_4(z_iy_{i+1}); \text{ for } 1 \leq i \leq n\} = 9n+3i+2 \text{ and i } \epsilon \text{ odd number} \\
W_{f_4} &= \{w_{f_4} + f_4(z_iy_{i+1}); \text{ for } 1 \leq i \leq n\} = 6n+3i+3 \text{ and i } \epsilon \text{ even number} \\
W_{f_4} &= \{w_{f_4} + f_4(y_jy_{j+1}); \text{ jika } 1 \leq j \leq m\} = 6n+3j+2
\end{aligned}$$

It is not difficult to see that the set $W_{f_4} = \{6n+4,\ 6n+5,\ \ldots, 12n+2\}$ contains an arithmetic sequence with the first term $8n+6$ and common difference 1. Thus α_3 is a super $(6n+4, 1)$-edge-antimagic total labeling. This concludes the proof. □

Conclusion

We can conclude that the graph $\mathbb{F}\ _n$ admit a super (a, d)-edge-antimagic total labeling for all feasible d and $n \geq 1$.

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