ON THE DOMINATION NUMBER OF SOME FAMILIES OF SPECIAL GRAPHS

Ika Hesti Agustin[1,2], Dafik[1,3]
[1]CGANT-Universitas Jember
[2]Department of Mathematics FMIPA University of Jember
Hestyarin@gmail.com
[3]Department of Mathematics Education FKIP University of Jember,
d.dafik@unej.ac.id

Abstract

A domination in graphs is part of graph theory which has many applications. Its application includes the morphological analysis, computer network communication, social network theory, CCTV installation, and many others. A set D of vertices of a simple graph G, that is a graph without loops and multiple edges, is called a dominating set if every vertex $u \in V(G) - D$ is adjacent to some vertex $v \in D$. The domination number of a graph G, denoted by $\gamma(G)$, is the order of a smallest dominating set of G. A dominating set D with $|D| = \gamma(G)$ is called a minimum dominating set, see Haynes and Henning [5] . This research aims to find the domination number of some families of special graphs, namely Spider Web graph Wb_n, Helmet graph $H_{n,m}$, Parachute graph Pc_n, and any regular graph. The results shows that the resulting domination numbers meet the lower bound of an obtained lower bound $\gamma(G)$ of any graphs.

Key Words *: dominating set, dominating number, special graphs.*

Introduction

A domination in graphs is part of graph theory which has many applications. Its application includes the morphological analysis, computer network communication, social network theory, CCTV installation, and many others [3]. The idea of domination in graphs arose in chessboard problems [12]. In 1862, de Jaenisch posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. A graph G may be formed by an 8×8 chessboard by taking the squares as the vertices with two vertices adjacent if a queen situated on one square attacks the other square. This problem can be modeled by finding a dominating set of five queens [6, 7].

All graphs in this paper are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. $|V(G)| = p$ and $|E(G)| = q$ are respectively number of vertices and edges. We refer the reader to [1] or [2] for all other terms and notation not provided in this paper. Haynes and Henning [5] defines that a set D of vertices of a simple graph G is called a dominating set if every vertex $u \in V(G) - D$ is adjacent to some vertex $v \in D$. The domination number of a graph G, denoted by $\gamma(G)$, is the order of a smallest dominating set of G [8, 9, 10]. A dominating set D with $|D| = \gamma(G)$ is called a minimum dominating set[13]. Haynes and Henning showed that

Theorem 1 *[5] Let $G = C_n$ be a cycle graph of order n and size n. It satisfies the following*

$$\gamma(C_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 (\mathrm{mod}\, 3) \,\mathrm{and}\, n \geq 6 \\ \lceil \frac{n}{3} \rceil + 1, & n \text{ otherwise} \end{cases}$$

Furthermore Murugesan proved the following theorem

Theorem 2 *[11] Let $G = (V, E)$ is a simple graph of order n. If G has a vertex v of degree $n-1$ then $\gamma(G) = 1$.*

This research aims to find the domination number of some families of special graphs of distance one, namely Spider Web graph Wb_n, Helmet graph $H_{n,m}$, Parachute graph Pc_n, and any regular graph. To determine the domination number of those graphs, we need a knowledge of a lower and upper bound of $\gamma(G)$.

A Useful Lemma

The following theorem gives a feasible $\gamma(G)$ for any graph. The theorem is very useful, therefore we reproof it for a hint to prove the resulting theorems of this research.

Theorem 3 *[4] Let $\gamma(G)$ be a domination number. For any graph G of order p, size q and maximum degrees $\Delta(G)$ will satisfies the following:*

$$\lceil \frac{p}{1+\triangle(G)} \rceil \leq \gamma(G) \leq p - \triangle(G)$$

Proof: Let S be a $\gamma - set$ of G. first we consider the lower bound. Each vertex can dominate at most itself and $\Delta(G)$ other vertices. Hence, $\gamma(G) \geq \lceil \frac{p}{1+\Delta(G)} \rceil$. For the upper bound, let v be a vertex of maximum degree $\Delta(G)$ and $N[v]$ be an out-neighbor vertices of v. Then v dominates $N[v]$ and the vertices in $V - N[v]$ dominate themselves. Hence, $V - N[v]$ is the dominating set of cardinality $n - \Delta(G)$, so $\gamma(G) \leq n - \Delta(G)$. It implies that $\lceil \frac{p}{1+\Delta(G)} \rceil \leq \gamma(G) \leq p - \triangle(G)$.□

The Results

The followings show $\gamma(G)$ when G are Spider Web graph Wb_n, Helmet graph $H_{n,m}$, Parachute graph Pc_n, and any regular graph. The steps for determining $\gamma(G)$ refer to the well-known technique, namely a pattern recognition. Trace for a smaller order of graph, then for a bigger one and finally for any order of graph. By tracing a pattern then we determine $\gamma(G)$ mathematically by using an arithmetic or a geometric sequence. Once we have established the theorem then we prove it deductively or inductively.

◇ **Teorema 1** *Let $G = Wb_n$ be a spider web graph. For $n \geq 3$, the domination number of G is*

$$\gamma(Wb_n) = \begin{cases} 2, & n = 3, 4 \\ \lceil \frac{n}{3} \rceil + 1, & n \text{ otherwise} \end{cases}$$

Proof. Let G be a spider web graph, denoted by Wb_n. The vertex set of G is $V(Wb_n) = \{x, x_i, y_i; 1 \leq i \leq n\}$ and the edge set is $E(Wb_n) = \{xx_i; 1 \leq i \leq n\} \cup \{x_n x_1, x_i x_{i+1};\ 1 \leq i \leq n-1\} \cup \{x_i y_i; 1 \leq i \leq n\} \cup \{y_n y_1,\ y_i y_{i+1}; 1 \leq i \leq n-1\}$. The order of the graph Wb_n is $p = |V| = 2n + 1$ and the size is $q = |E| = 4n$. Let D_1 and D_2 be the dominating set. We define $D_1 = \{x_2, y_4; \text{for}\, n = 4\}$ and $D_2 = \{x, y_{3i-1}; 1 \leq i \leq \lceil \frac{n}{3} \rceil\}$. It is obvious to see that D_1 and D_2 dominates all vertices of the spider web graph. By direct calculation we can obtain that $|D_1| = 2$, $|D_2| = \lceil \frac{n}{3} \rceil + 1$. According to Theorem 3, the lower and upper bound is: $\lceil \frac{p}{1+\triangle(Wb_n)} \rceil \le \gamma(Wb_n) \le p - \triangle(Wb_n)$, where $\triangle(Wb_n)$ is the largest degree of the vertices. It implies that $\lceil \frac{2n+1}{1+\triangle Wb_n} \rceil \le \gamma(Wb_n) \le 2n+1-\triangle(Wb_n)$. Since $\triangle(Wb_4) = 4$, it follows that $\lceil \frac{9}{5} \rceil \le \gamma(Wb_n) \le 5$. It is easy to see that $\gamma(Wb_4) = 2$ is in the resulting interval. Furthermore, since $\triangle(Wb_n) = n$, for $n \ge 5$. It follows that $\lceil \frac{2n+1}{n+1} \rceil \le \gamma(Wb_n) \le n+1$. It is easy to see that $\gamma(Wb_n) = \lceil \frac{n}{3} \rceil + 1$ is in the resulting interval. These complete the proof. □

◇ **Teorema 2** *Let $G = Pc_n$ be a parachute graph. For $n \ge 3$, the domination number $\gamma(Pc_n) = \lceil \frac{n}{3} \rceil + 1$*

Proof. Let G be a parachute graph, denoted by Pc_n. The vertex set of G is $V(Pc_n) = \{x_i, y_i, A; 1 \le i \le n\}$ and the edge set is $E(Pc_n) = \{y_n x_1, y_1 x_n, Ax_i; 1 \le i \le n\} \cup \{x_i x_{i+1}, y_i y_{i+1}; 1 \le_i \le n-1\}$. The order of the graph Pc_n is $p = |V| = 2n+1$ and the size is $q = |E| = 3n$. Let D be a dominating set of the parachute graph. We define $D = \{A, y_{3i-1}; 1 \le i \le \lceil \frac{n}{3} \rceil\}$. It is obvious to verify that all elements of D dominates all vertices of the parachute graph. The cardinality of $|D| = \lceil \frac{n}{3} \rceil + 1$. According Theorem 3, the lower and upper bound is: $\lceil \frac{p}{1+\triangle Pc_n} \rceil \le \gamma(Pc_n) \le p - \triangle(Pc_n)$, where $\triangle(Pc_n)$ is the largest degree of the vertices. Since $\triangle(Pc_n) = n$, for $n \ge 3$, it follows that $\lceil \frac{2n+1}{n+1} \rceil \le \gamma(Pc_n) \le n+1$. It is easy to see that $\gamma(Pc_n) = \lceil \frac{n}{3} \rceil + 1$ is in the resulting interval, it completes the proof. □

◇ **Teorema 3** *Let $G = H_{n,m}$ be a helmet graph. For $n \ge 3$ and $m \ge 1$, the domination number $\gamma(H_{n,m}) = n$.*

Proof. Let G be a helmet graph, denoted by $H_{n,m}$. The vertex set of G is $V(H_{n,m}) = \{A, x_i, x_{i,j}; 1 \le i \le n, 1 \le j \le m\}$ and $E(H_{n,m}) = \{x_n x_1, Ax_i, x_i x_{i,j}; 1 \le i \le n, 1 \le j \le m\} \cup \{x_i x_{i+1}; 1 \le i \le n-1\}$. The order of the graph $H_{n,m}$ is $p = |V| = n(m+1)$ and the size is $q = |E| = n(m+2)$. Let D be a dominating set of the Helmet graph. We define $D = \{x_i;\ 1 \le i \le n\}$. It is easy to verify that D dominates all the vertices of the Helmet graph, and $|D| = n$. According Theorem 3, the lower and upper bound are stated as follow: $\lceil \frac{p}{1+\triangle H_{n,m}} \rceil \le \gamma(H_{n,m}) \le p - \triangle H_{n,m}$ where $\triangle H_{n,m}$ is the largest degree of the vertices. It implies that $\lceil \frac{n(m+1)}{1+\triangle H_{n,m}} \rceil \le \gamma(H_{n,m}) \le n(m+1) - \triangle H_{n,m}$. Since $\triangle H_{n,m} = n$, for $n \ge 3$ and $m \ge 1$, it follows that $\lceil \frac{n(m+1)}{n+1} \rceil \le \gamma(H_{n,m}) \le nm$. It is easy to see that $\gamma(H_{n,m}) = n$ is in the resulting interval. It completes the proof. □

◇ **Teorema 4** *Let $G = A_{2n,m}$ be any regular graph of order $2n$ and degree m. For $n \ge 3$ and $3 \le m < 2n$, the domination number $\gamma(A_n) = \lceil \frac{2n}{m+1} \rceil$*

Proof. Let G be a regular graph of order $2n$ and degree m, denoted by $A_{2n,m}$. The vertex set of G is $V(A_{2n,m}) = \{x_i, y_i; 1 \le i \le n\}$ and the edge set is $E(A_{2n,m}) = \{x_n x_1, y_n y_1, x_i x_{i+1}, y_i y_{i+1}; 1 \le i \le n-1\} \cup \{x_i y_{i+k(\mathrm{mod}\, n)}; 1 \le i \le$

$n\}$ and $1 \leq k \leq m-3$. The order of the graph $p = |V| = 2n$ and the size is $q = |E| = n(m+2)$. According Theorem 3, the lower and upper bound are stated as follow: $\lceil \frac{p}{1+\triangle A_{2n,m}} \rceil \leq \gamma(A_{2n,m}) \leq p - \triangle A_{2n,m}$ where $\triangle A_{2n,m}$ is the largest degree of the vertices. It implies that $\lceil \frac{2n}{1+\triangle A_{2n,m}} \rceil \leq \gamma(A_{2n,m}) \leq 2n - \triangle A_{2n,m}$. Since $\triangle A_{2n,m} = m$ with $3 \leq m \leq 2n$, it follows that $\lceil \frac{2n}{m+1} \rceil \leq \gamma(A_{2n,m}) \leq 2n - m$. It can be seen that $\gamma(A_{2n,m}) = \lceil \frac{2n}{m+1} \rceil$ is exactly the lower bound. $\square$

Conclusions

We have studied the domination number of some families of special graphs of distance one, namely Spider Web graph Wb_n, Helmet graph $H_{n,m}$, Parachute graph Pc_n, and any regular graph. The results obtain that all the values of $\gamma(G)$ take a place in the interval $\lceil \frac{p}{1+\triangle(G)} \rceil \leq \gamma(G) \leq p - \triangle(G)$. The result shows that for $n \geq 3, \gamma(Wb_n), \gamma(Pc_n), \gamma(H_{n,m})$ and $\gamma(A_{2n,m})$ are respectively as follows:

$$\gamma(Wb_n) = \begin{cases} 2, & n = 4 \\ \lceil \frac{n}{3} \rceil + 1, & n \text{ otherwise} \end{cases}$$

$$\gamma(Pc_n) = \lceil \frac{n}{3} \rceil + 1$$

$$\gamma(H_{n,m}) = n \text{ for } m \geq 1$$

$$\gamma(A_{2n,m}) = \lceil \frac{2n}{m+1} \rceil \text{ for } 3 \leq m < 2n$$

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