Super (a,d)-edge-antimagic total labeling of connected Disc Brake graph

Inge Yosanda Arianti[2], Dafik[1,2], Slamin[1,2]
[1]CGANT- University of Jember
[2]Department of Mathematics Education FKIP University of Jember,
ingeyosandaarianti, d.dafik@gmail.com
[3]Department of Information System Universitas Jember,
s.slamin@gmail.com

Abstract

Super edge-antimagic total labeling of a graph $G = (V, E)$ with order p and size q, is a vertex labeling $\{1, 2, 3, ...p\}$ and an edge labeling $\{p+1, p+2, ...p+q\}$ such that the edge-weights, $w(uv) = f(u)+f(v)+f(uv), uv \in E(G)$ form an arithmetic sequence and for $a > 0$ and $d \geq 0$, where $f(u)$ is a label of vertex u, $f(v)$ is a label of vertex v and $f(uv)$ is a label of edge uv. In this paper we discuss about super edge-antimagic total labelings properties of connective Disc Brake graph, denoted by $Db_{n,p}$. The result shows that a connected Disc Brake graph admit a super (a, d)-edge antimagic total labeling for $d = 0, 1, 2$, $n \geq 3$, n is odd and $p \geq 2$. It can be concluded that the result has covered all the feasible d.

Key Words : *(a, d)-edge-antimagic total labeling, super (a, d)-edge-antimagic total labeling, Disc Brake graph.*

Introduction

Mathematics is one of basic science which play an important role on the real life. Mathematics is always needed in the advance of modern technology. Mathematics consists of several branches, one of them is a discrete mathematics. One of an interesting topic in discrete mathematics is graph theory. There are many topic in graph theory, such as colouring, graph labelings etc. For more detail of basic definition of graph can be found in [7],[9],[14],[18].

In this paper we will focus to the one of interesting topic in graph theory, namely graph labeling. A labeling of a graph is any mapping that sends some set of graph elements to a set of positive integers. If the domain is the vertex-set or the edge-set, the labelings are called, respectively, vertex labeling or edge labeling. Moreover, if the domain is $V(G) \cup E(G)$ then the labelings are called *total* labelings. We define the *edge-weight* of an edge $uv \in E(G)$ under a total labeling to be the sum of the vertex labels corresponding to vertices u, v and edge label corresponding to edge uv. If such a labeling exists then G is said to be an *(a, d)-edge-antimagic total graph*. Such a graph G is called *super* if the smallest possible labels appear on the vertices. Thus, a *super (a, d)-edge-antimagic total graph* is a graph that admits a super (a, d)-edge-antimagic total labeling. For more detail about the basic definition of a graph labeling can be found in [8],[13], and [16].

These labelings, introduced by Simanjuntak *at al.* in [17], are natural extensions of the concept of magic valuation, studied by Kotzig and Rosa [1]. Now the magic labelings give an extension of many type of labelings. Such, vertex antimagic total labelings, edges antimagic total labelings and super vertex or edges antimagic total labelings. Dafik in [3, 4, 5, 12] has found some research about graph labelings, such as super (a, d)-edge-antimagic total labelings, namely mK, n, n, on antimagic labellings of disjoint union of complete s-partite graph and edge-antimagic total labelling of disjoint union caterpillars. And also Lee, Ming-ju has found On Super (a,1)-edge Antimagic Total Labelings Subdifition stars in [11]. This paper will discussed about super (a, d)-edge-antimagic total labeling of connected Disc Brake graph.

Some Useful Lemmas

We start our result by a necessary condition for a graph to be super (a, d)-edge antimagic, providing a least upper bound for feasible values of d. The first lemma can be found in [10].

Lemma 1 *If a (p, q)-graph is super (a, d)-edge antimagic total then* $d \leq \frac{2p+q-5}{q-1}$.

Proof. Assume that a (p, q)-graph has a super (a, d)-edge antimagic total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ and the edge-weights $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$. The minimum possible edge-weight in the labeling f is at least $1+2+p+1 = p+4$. Thus, $a \geq p+4$. On the other hand, the maximum possible edge-weight is at most $(p-1)+p+(p+q) = 3p+q-1$. So we obtain $a+(q-1)d \leq 3p+q-1$ which gives the desired upper bound for the difference d. □

The second lemma obtainded by Figueroa-Centeno et al [15], gives a necessary and sufficient condition for a graph to be super $(a, 0)$-edge-antimagic total

Lemma 2 A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p+q+s$, where $s = min(S)$ and $S = \{a-(p+1), a-(p+2), \ldots, a-(p+q)\}$.

The third lemma obtainded by Robiyatul adawiyah in [2].

Lemma 3 *Let Ψ be a set of consecutive integers $\Psi = \{c, c+1, c+2, \ldots, c+k\}$, with k is even. There will be a permutation $\Pi(\Psi)$ from element Ψ so $\Psi + \Pi(\Psi)$ forms consecutive integers $\Psi+\Pi(\Psi) = \{2c+\frac{k}{2}, 2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, \ldots, 2c+\frac{3k}{2}\}$.*

Proof. Let Ψ be a sequence $\Psi = \{v_i | v_i = c + (i-1), 1 \le i \le k+1\}$ and k is even, defined the permutation $\Pi(\Psi) = \{w_i | 1 \le i \le k+1\}$ of the element Ψ as follows:

$$w_i = \begin{cases} c + i + \frac{k}{2} - 1, & \text{if } 1 \le i \le \frac{k}{2} + 1 \\ c + i - (\frac{k}{2} + 2), & \text{if } \frac{k}{2} + 2 \le i \le k + 1 \end{cases}$$

For $1 \le i \le \frac{k}{2} + 1$, $\bigcup_{i=1}^{\frac{k}{2}+1} w_i = \{c + \frac{k}{2}, c + \frac{k}{2} + 1, \ldots, c + k - 1\ c + k\}$

For $\frac{k}{2} + 2 \le i \le k + 1$,$\bigcup_{i=\frac{k}{2}+2}^{k+1} w_i = \{c, c + 1, \ldots, c + \frac{k}{2} - 2\ c + \frac{k}{2} - 1\}$

By direct computation, we obtain that:
$\Psi + \Pi(\Psi) = \{v_i + w_i | 1 \le i \le k+1\} = \{2c + 2i + \frac{k}{2} - 2\ | 1 \le i \le \frac{k}{2} + 1\} \cup \{2c + 2i - \frac{k}{2} - 3 \mid \frac{k}{2} + 2 \le i \le k + 1\} = \{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, \ldots, 2c + \frac{3k}{2}\}$

Previously, the lemma states that a (p, q)-graph G is super $(a, 0)$-edge-antimagic total if and only if there exists an (a-p-q; 1)-edge-antimagic vertex labeling. The two above lemma will be used for develop theorem 1.

The Results of Research

Disc Brake graph denoted by $Db_{n,p}$ with vertex set $V(Db_{n,p}) = \{x_{i,j}; 1 \le i \le n\ , 1 \le j \le p\} \cup \{y_{k,l}; 1 \le k \le 2n\ , 1 \le l \le p-1\}$. $E(Db_{n,p}) = \{x_{i,j}x_{i+1,j}; 1 \le i \le n\ , 1 \le j \le p\} \cup \{x_{n,j}x_{1,j}\} \cup \{y_{k,l}y_{k+1,l}; k\ ganjil; 1 \le l \le p-1\} \cup \{x_{i,j}y_{2i-2,j}; 2 \le i \le n\ , 1 \le j \le p-1\} \cup \{x_{1,j}y_{2n,j}; 1 \le j \le p-1\} \cup \{x_{i,j}y_{2i-1,j}; 2 \le i \le n\, 1 \le j \le p-1\} \cup \{x_{1,j}y_{1,j}; 1 \le j \le p-1\} \cup \{x_{i,j}y_{2i-2,j}; 2 \le i \le n\ , 2 \le j \le p-1\} \cup \{x_{1,j}y_{2n,j-1}; 2 \le j \le p-1\} \cup \{x_{i,j}y_{2i-1,j}; 2 \le i \le n\, 2 \le j \le p-1\} \cup \{x_{1,j}y_{1,j-1}; 2 \le j \le p-1\}$. Thus $|V(Db_{n,p})| = p = 3np - 2n$ and $|E(Db_{n,p})| = q = 6np - 5n$.

If Disc Brake graph has a super (a, d)-edge-antimagic total labeling then, for $p = 3np - 2n$ and $q = 6np - 5n$, it follows from Lemma 1 that the upper bound of d is $d \le 2$ or $d \in \{0, 1, 2\}$. The following lemma describes an $(a, 1)$-edge-antimagic vertex labeling for Disc Brake graph.

Prior to find the main theorem of the existence property of SEATL of Disc Brake graph, we propose the following lemma:

◇ **Teorema 1** *The graph $Db_{n,p}$ has an $(3,1)$-edge-antimagic vertex labeling for odd $n \geq 3$ and $p \geq 2$.*

Proof. Define the vertex labeling $f_1 : Db_{n,p} \rightarrow \{1, 2, \ldots, 3np - 2n\}$ in the following way:

$$
\begin{aligned}
f_1^1(x_{i,j}) &= 3nj - 3n + \frac{i+1}{2}, \text{ untuk } 1 \leq i \leq n,\ 1 \leq j \leq p, \text{dan i ganjil} \\
f_1^2(x_{i,j}) &= 3nj - 3n + \frac{i+1+n}{2}, \text{ untuk } 1 \leq i \leq n,\ 1 \leq j \leq p, \text{dan i genap} \\
f_1^3(y_{k,l}) &= 3nl - 2n + \frac{k+2}{4}, \text{ untuk } 1 \leq i \leq n,\ 1 \leq j \leq p, \text{dan i genap} \\
f_1^4(y_{k,l}) &= 3nl - 2n + \frac{2n+k+2}{4}, \text{ untuk } 1 \leq k \leq 2n,\ 1 \leq l \leq p-1, \\
&\quad k = 4(mod\ 4) \\
f_1^5(y_{k,l}) &= 3nl - n + \frac{k+1}{4}, \text{ untuk } 1 \leq k \leq 2n,\ 1 \leq l \leq p-1, \\
&\quad k = 3(mod\ 4) \\
f_1^6(y_{k,l}) &= 3nl - n + \frac{2n+k+1}{4}, \text{ untuk } 1 \leq k \leq 2n,\ 1 \leq l \leq p-1, \\
&\quad k = 1(mod\ 4)
\end{aligned}
$$

The vertex labeling f_1 is a bijective function. The edge-weights of $Db_{n,p}$, under the labeling f_1, constitute the following sets:

$$
\begin{aligned}
w_{f_1}^1(x_{n,j}x_{1,j}) &= 6nj - 6n + \frac{n+3}{2} \\
w_{f_1}^2(x_{i,j}x_{i+1,j}) &= 6nj - 6n + \frac{n+3}{2} + i \\
w_{f_1}^3(x_{1,j}y_{2n,j}) &= 6nj - 5n + \frac{n+3}{2} \\
w_{f_1}^4(x_{i,j}y_{2i-2,j}) &= 6nj - 5n + \frac{n+3}{2} + i - 1 \\
w_{f_1}^5(x_{1,j}y_{1,j}) &= 6nj - 4n + \frac{n+3}{2} \\
w_{f_1}^6(x_{i,j}y_{2i-1,j}) &= 6nj - 4n + \frac{n+3}{2} + i - 1 \\
w_{f_1}^7(y_{k,l}y_{k+1,l}) &= 6nl - 3n + \frac{n+3}{2} + \frac{k+1}{2} - 1 \\
w_{f_1}^8(x_{1,j}y_{2n,j-1}) &= 6nj - 2n + \frac{n+3}{2} \\
w_{f_1}^9(x_{i,j}y_{2i-2,j}) &= 6nj - 2n + \frac{n+3}{2} + i - 1
\end{aligned}
$$

$$\begin{aligned}
w^{10}_{f_1}(x_{1,j}y_{1,j-1}) &= 6nj - n + \frac{n+3}{2} \\
w^{11}_{f_1}(x_{i,j}y_{2i-1,j}) &= 6nj - n + \frac{n+3}{2} + i - 1
\end{aligned}$$

From the formula of edge-weights above, we can see that the set $\bigcup_{t=1}^{11} w^t_{f_1}$ $=\{3, 4, 5, \ldots, 6np - 6n + \frac{n+3}{2} + n - 1\}$ consists of consecutive integers. Thus f_1 is a $(3, 1)$-edge antimagic vertex labeling. □

With Theorem 1 in hand together with Lemma 2, we can establish the following theorem.

◇ **Teorema 2** *The graph $Db_{n,p}$ has a super $(9np - 7n + \frac{n+3}{2}, 0)$-edge-antimagic total labeling $n \geq 3$,n is odd and $p \geq 2$.*

◇ **Teorema 3** *The graph $Db_{n,p}$ has a super $(\frac{n+3}{2}+3np-2n+1, 2)$-edge-antimagic total labeling $n \geq 3$,n is odd and $p \geq 2$.*

Proof. For $d = 2$. The *edge* label of $Db_{n,p}$ for $d = 2$ are:

$$\begin{aligned}
f^1_2(x_{n,j}x_{1,j}) &= 3np - 8n + 6nj + 1 \\
f^2_2(x_{i,j}x_{i+1,j}) &= 3np - 8n + 6nj + i + 1 \\
f^3_2(x_{1,j}y_{2n,j}) &= 3np - 7n + 6nj + 1 \\
f^4_2(x_{i,j}y_{2i-2,j}) &= 3np - 7n + 6nj + i \\
f^5_2(x_{1,j}y_{1,j}) &= 3np - 6n + 6nj + 1 \\
f^6_2(x_{i,j}y_{2i-1,j}) &= 3np - 7n + 6nj + 1 \\
f^7_2(y_{k,l}y_{k+1,l}) &= 3np - 5n + 6nl + \frac{k+1}{2} \\
f^8_2(x_{1,j}y_{2n,j}) &= 3np - 4n + 6nj + 1 \\
f^9_2(x_{i,j}y_{2i-2,j}) &= 3np - 4n + 6nj + i \\
f^{10}_2(x_{1,j}y_{1,j-1}) &= 3np - 3n + 6nj + 1 \\
f^{11}_2(x_{i,j}y_{2i-1,j}) &= 3np - 3n + 6nj + i
\end{aligned}$$

The total labeling f_2 is a bijective function from $V(Db_{n,p}) \cup E(Db_{n,p})$. The edge-weights of $Db_{n,p}$ can be defined as follow:

$$
\begin{aligned}
W^1_{f_2}(x_{n,j}x_{1,j}) &= 12nj - 14n + 3np + \frac{n+3}{2} + 1\\
W^2_{f_2}(x_{i,j}x_{i+1,j}) &= 12nj - 14n + 3np + \frac{n+3}{2} + 2i + 1\\
W^3_{f_2}(x_{1,j}y_{2n,j}) &= 12nj - 12n + 3np + \frac{n+3}{2} + 1\\
W^4_{f_2}(x_{i,j}y_{2i-2,j}) &= 12nj - 12n + 3np + \frac{n+3}{2} + 2i - 1\\
W^5_{f_2}(x_{1,j}y_{1,j}) &= 12nj - 10n + 3np + \frac{n+3}{2} + 1\\
W^6_{f_2}(x_{i,j}y_{2i-1,j}) &= 12nj - 10n + 3np + \frac{n+3}{2} + 2i - 1\\
W^7_{f_2}(y_{k,l}y_{k+1,l}) &= 12nl - 8n + 3np + \frac{n+3}{2} + k\\
W^8_{f_2}(x_{1,j}y_{2n,j-1} &= 12nj - 6n + 3np + \frac{n+3}{2} + 1\\
W^9_{f_2}(x_{i,j}y_{2i-2,j}) &= 12nj - 6n + 3np + \frac{n+3}{2} + 2i - 1\\
W^{10}_{f_2}(x_{i,j}y_{2i-2,j}) &= 12nj - 4n + 3np + \frac{n+3}{2} + 1\\
W^{11}_{f_2}(x_{i,j}y_{2i-1,j}) &= 12nj - 4n + 3np + \frac{n+3}{2} + 2i - 1
\end{aligned}
$$

From the first and second case we can conclude that if the graph $Db_{n,p}$ has a super ($\frac{n+3}{2} + 9np - 7n$,0)-edge-antimagic total labeling and a super ($\frac{n+3}{2} + 3np - 2n + 1, 2$)-edge-antimagic total labeling for $n \geq 3$, n is odd and $p \geq 2$. $\square$

◇ **Teorema 4** *The graph $Db_{n,p}$ has a super $(6np - 4n + 2, 1)$-edge-antimagic total labeling $n \geq 3$,n is odd and $p \geq 2$.*

Proof. From lemma 4, the graph $Db_{n,p}$ has a $(3, 1)$-edge-antimagic vertex labeling. This means that the graph $Db_{n,p}$ has a set of hand weights are expressed in $\{5, 6, 7, \ldots, 6np - 5n\}$, in other words the graph $Db_{n,p}$ have a row of hand weights with initial values $a = 5$ and different from each tribe is 1. If we let the row weights $Db_{n,p}$ stated in $\Upsilon = \{c, c + 1, c + 2, \ldots, c + k\}$ the obtained value $c = 5$ and $k = q - 1 = 6np - 5n - 1$. In light of lemma 3, there exists a permutation $\Pi(\Psi)$ of the elements of Ψ such that $\Psi + [\Pi(\Psi) - c + p + 1]\ \Upsilon$ is the weight the total of these functions.

$$\begin{aligned}
\Upsilon + (\Pi(\Upsilon) + \eta) &= a \\
c + (c + 1 + \frac{k}{2} - 1) + \eta &= 6np - 4n + 2 \\
2c + \frac{k}{2} + \eta &= 6np - 4n + 2 \\
10 + \frac{k}{2} + \eta &= 6np - 4n + 2 \\
\eta &= 6np - 4n - 8 - \frac{k}{2}
\end{aligned}$$

So $\Upsilon + (\Pi(\Upsilon) + \eta)$ is the total weight of the function. Proving the lemma 4 has been mentioned a total weight smallest in $i = 1$, so:

$$\begin{aligned}
\Upsilon + (\Pi(\Upsilon) + \eta) &= c + (c + i + \frac{k}{2} - 1 + 6np - 4n - 8 - \frac{k}{2}) \\
&= 2c + \frac{k}{2} + 6np - 4n - 8 - \frac{k}{2} \\
&= 6np - 4n + 2
\end{aligned}$$

Second smallest total weight in $i = \frac{k}{2} + 2$:

$$\begin{aligned}
\Upsilon + (\Pi(\Upsilon) + \eta) &= c + \frac{k}{2} + 1 + c + i - \frac{k}{2} - 2 + 6np - 4n - 8 - \frac{k}{2} \\
&= 2c + 1 + \frac{k}{2} + 6np - 4n - 8 - \frac{k}{2} \\
&= 6np - 4n + 3
\end{aligned}$$

And so on, up to the total weight of the biggest in $i = \frac{k}{2} + 1$:

$$\begin{aligned}
\Upsilon + (\Pi(\Upsilon) + \eta) &= c + c + i - 1 + 12np - 9n - 9 - \frac{k}{2} \\
&= 2c + \frac{k}{2} + 1 - 1 + 12np - 9n - 9 - \frac{k}{2} \\
&= 2c - 9n + 12np - 9 \\
&= 12np - 9n + 1
\end{aligned}$$

Based on the above calculation are according to lemma 3, there exists a permutation $\Pi(\Psi)$ of the elements of Ψ such that $\Psi + [\Pi(\Psi) - c + p + 1] =$

$\{6np-4n+2, 6np-4n+3, \ldots, 12np-9n+1\}$. If $[\Pi(\Psi)-c+p+1]$ is an edge labeling of $Db_{n,p}$ for $n \geq 3$, n is odd and $p \geq 2$,then $\Psi+[\Pi(\Psi)-c+p+1]$ determines the set of edge-weight of the graph $Db_{n,p}$ and the result total labeling is super $(6np-4n+2,1)$-edge-antimagic total. This concludes the proof.

Conclusion

Based on the result above we can conclude that the Disc Brake graph admit a super (a,d)-edge antimagic total labeling for all feasible d, $n \geq 3$,n is odd and $p \geq 2$ with $d \in \{0,1,2\}$. The natural question is, if graph G is SEATL, is the disjoint of graph G SEATL as well? To answer the question we propose the following open problem for graph $G = Db_{n,p}$.

Open Problem 1 *For $n \geq 3$,n is odd and $p \geq 2$, determine if there exists a super (a,d)-edge-antimagic total labeling of disjoint $Db_{n,p}$ with any feasible upper bound d.*

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