On the edge r-dynamic chromatic number of some related graph operations

Novian Nur Fatihah[1,3], Arika Indah Kriatiana [1,2], Ika Hesti Agustin [1,3], Dafik[1,2]
[1]CGANT University of Jember Indonesia
[2]Mathematics Depart. University of Jember Indonesia [3]Mathematics Edu. Depart. University of Jember Indonesia
e-mail: novianfatihah@yahoo.com

Abstract—All graphs in this paper are simple, nontrivial, connected and undirected. By an edge proper k-coloring of a graph G, we mean a map c : $E(G) \to S$, where $|S| = k$, such that any two adjacent edges receive different colors. An edge r-dynamic k-coloring is a proper k-coloring c of G such that $|c(N(uv))| \geq min\ (r, d(u) + d(v) - 2)$ for each edge uv in $V(G)$, where $N(uv)$ is the neighborhood of uv and $c(S) = c(uv) : uv2S$ for an edge subset S. The edge r-dynamic chromatic number, written as $\lambda_r(G)$, is the minimum k such that G has an edge r-dynamic k-coloring. In this paper, we will determine the edge coloring r-dynamic number of a comb product of some graph, denote by $G \trianglerighteq H$. Comb product of some graph is a graph formed by two graphs G and H, where each edge of graph G is replaced by which one edge of graph H.

Keywords—r-dynamic chromatic number, graph coloring, exponential graphs

INTRODUCTION

Graph theory is part of discrete mathematics which is mostly studied by researchers. This is due to the wide of application in real life. One of the theory developed in graph theory is the coloring. The new extension of graph color is r-dynamic coloring. The r-dynamic chromatic number, introduced by Montgomery [1] and written as $\chi_r(G)$, is the least k such that G has an r-dynamic k-coloring. Note that the 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number of graph has been studied under the name a dynamic chromatic number, denoted by $\chi_d(G)$. In [1], he conjectured $\chi_2(G) \leq \chi(G)+2$ when G is regular, which remains open. Akbari *et.al.* [2] proved Montgomery's conjecture for bipartite regular graphs, as well as Lai, *et.al.* [3] proved $\chi_2(G) \leq \Delta(G)+1$ for $\Delta(G) \leq 3$ when no component is the 5-cycle.

By a greedy coloring algorithm, Jahanbekama [4] proved that $\chi_r(G) \leq r\Delta(G) + 1$, and equality holds for $\Delta(G) > 2$ if and only if G is r-regular with diameter 2 and girth 5. They improved the bound to $\chi_r(G) \leq \Delta(G) + 2r - 2$ when $\delta(G) > 2r \ln n$ and $\chi_r(G) \leq \Delta(G)+r$ when $\delta(G) > r^2 \ln n$.

The following observation is useful to find the exact values of r-dynamic chromatic number.

Observation 1. *Let $\delta(G)$ and $\Delta(G)$ be a minimum and maximum degree of a graph G, respectively. Then the followings hold*

- $\chi_r(G) \geq \min\{\Delta(G), r\} + 1,$
- $\chi(G) \leq \chi_2(G) \leq \chi_3(G) \leq \cdots \leq \chi_{\Delta(G)}(G),$
- $\chi_{r+1}(G) \geq \chi_r(G)$ *and if* $r \geq \Delta(G)$ *then* $\chi_r(G) = \chi_{\Delta(G)}(G)$.

There are some researches studied this problem, some of them can be found in [5],[6],[7],[8],[9].

THE RESULTS

We are ready to show our main theorems. In this study we used exponential graph, there are two theorems found in this study. These deals with exponential graph $C_n \trianglerighteq P_2$ and $C_n \trianglerighteq C_m$.

Theorem 1. *Let G be a comb product denote by $C_n \trianglerighteq P_2$ for $n \geq 3$, the vertex r-dynamic chromatic number is:*

$$\chi(C_n \trianglerighteq P_2) = \begin{cases} 3, & n \text{ even} \\ 3, & n \text{ odd} \end{cases}$$

$$\chi_d(C_n \trianglerighteq P_2) = \begin{cases} 3, & n = 3k, k \in N \\ 4, & n \text{ otherwise} \end{cases}$$

$$\chi_{r\geq 3}(C_n \trianglerighteq P_2) = \begin{cases} 4, & n = 3k, n = 4k, k \in N \\ 4, & n \text{ otherwise} \\ 5, & n = 5k, k \in N \end{cases}$$

Proof. A comb product of graph $C_n \trianglerighteq P_2$, for $n \geq 3$, is a connected graph with vertex set $V(C_n \trianglerighteq P_2) = \{x_i, 1 \leq i \leq n\} \cup \{y_i, 1 \leq i \leq n\}$, and edge set $E(C_n^{P_2}) = \{x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{x_iy_i; 1 \leq i \leq n\}$. The order and size of $C_n \trianglerighteq P_n$,$n \geq 3$ are $|V(C_n \trianglerighteq P_n)| = 2n$ and $|E(C_n \trianglerighteq P_2)| = 2n$. A comb product graph
$C_n \trianglerighteq P_2$ is regular graph of degree 3, thus $C_n \trianglerighteq P_2$,
$\delta(C_n \trianglerighteq P_2) = \Delta(C_n \trianglerighteq P_2) = 3$. By observation 1, $\chi_r(C_n \trianglerighteq P_2) \geq \min\{\Delta(C_n \trianglerighteq P_2), r\} = \min\{3, r\}$. To find the exact value of r-dynamic chromatic number of $C_n \trianglerighteq P_2$, we define three cases, namely $\chi(C_n \trianglerighteq P_2)$, $\chi_2(C_n \trianglerighteq P_2)$ and $\chi_{r\geq 3}(C_n \trianglerighteq P_2)$.

For $r = 1$, the lower bound $\chi(C_n \trianglerighteq P_2) \geq \min\{3, 1\} = 1$, and for $r = 2$, the lower bound $\chi(C_n \trianglerighteq P_2) \geq \min\{3, 2\} = 2$. We will prove that $\chi(C_n \trianglerighteq P_2) \leq 3$ by defining a map $c_1 : V(C_n \trianglerighteq P_2) \to \{1, 2, \dots, k\}$ for $n \geq 3$, by the following:

$$12 \dots 12,\ n \text{ even}$$
$$123 \dots 123,\ n \text{ odd}$$

$$c_1(y_1, y_2, \dots, y_n) = \begin{cases} 21 \dots 21, & n \text{ even} \\ 212 \dots 212, & n \text{ odd} \end{cases}$$

It is easy to see that c_1 gives $\chi(C_n \trianglerighteq P_2) \leq 2$ for n even, but for n odd, we could not avoid to have $\chi(C_n \trianglerighteq P_2) \leq 3$, otherwise there are at least two adjacent vertices assigned the same colors. Thus $\chi(C_n \trianglerighteq P_2) = 2$ for n even and $\chi(C_n \trianglerighteq P_2) \leq 3$, for n odd.

For $\chi_r(C_n \trianglerighteq P_2)$ and $r \geq 3$, the lower bound $\chi_3(C_n \trianglerighteq P_2) \geq \min\{3, 3\} = 3$. We will prove that $\chi_3(C_n \trianglerighteq P_2) \leq 4$ by defining a map $c_3 : V(C_n \trianglerighteq P_2) \to \{1, 2, \dots, k\}$ for $n \geq 3$, by the following.

$$c_3(x_1, x_2, \dots, x_n) = \begin{cases} 123 \dots 123, & n = 3k, \\ 1234 \dots 1234, & n = 4k, \\ 12345 \dots 12345, & n = 5k, \\ 1231234, & n = 7, n \equiv 7 \pmod 3, \\ 12341234123, & n = 11, \end{cases}$$

$$c_3(y_1, y_2, \ldots, y_n) = \begin{cases} 444 \ldots 444,\ n = 3k, \\ 3412 \ldots 3412, n = 4k, \\ 45123 \ldots 45123, \\ n = 5k, \\ 3444412,\ n = 7, \\ n \equiv 7(\text{mod } 3), \\ 44123412344,\ n = 11, \end{cases}$$

It is easy to see that c_3 gives $\chi_3(C_n \trianglerighteq P_2) \leq 4$, for $n = 3k, n = 4k, k \in N$, but for $n = 5k$ we are forced to have $\chi_3(C_n \trianglerighteq P_2) = 5$ as well as $\chi_3(C_n \trianglerighteq P_2) = 4$ for n otherwise.
Thus $\chi_3(C_n \trianglerighteq P_2) = 4$, for $n = 4k$, and $\chi_3(C_n \trianglerighteq P_2) = 5$ for n otherwise. By observation 1 $r \geq \Delta(C_n \trianglerighteq P_2) = 3$, it immediately gives $\chi_3(C_n \trianglerighteq P_2)$=$\chi_r(C_n \trianglerighteq P_2)$ for $n \geq 3$. □

Theorem 2. *Let G be a comb product denote by $C_n \trianglerighteq P_2$ for $n \geq 3$, the edges r-dynamic chromatic number is:*

$$\lambda(C_n \trianglerighteq P_2) = \lambda_d(C_n \trianglerighteq P_2) = \lambda_3(C_n \trianglerighteq P_2) = 3$$

$$\lambda_{r\geq 3}(C_n \trianglerighteq P_2) = 4$$

Proof. A comb product of graph $C_n \trianglerighteq P_2$, for $n \geq 3$, is a connected graph with vertex set $V(C_n \trianglerighteq P_2) = \{x_i, 1 \leq i \leq n\} \cup \{y_i, 1 \leq i \leq n\}$, and edge set $E(C_n \trianglerighteq P_2) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i y_i; 1 \leq i \leq n\}$. The order and size of $C_n \trianglerighteq P_n$,$n \geq 3$ are $|V(C_n \trianglerighteq P_n)| = 2n$ and $|E(C_n \trianglerighteq P_2)| = 2n$. A comb product of graph $C_n \trianglerighteq P_2$ is regular graph of degree 3, thus $C_n \trianglerighteq P_2$, $\delta(C_n \trianglerighteq P_2) = \Delta(C_n \trianglerighteq P_2) = 3$. By observation when $\chi_r(C_n \trianglerighteq P_2) \geq \min\{\Delta(C_n \trianglerighteq P_2), r\} = \min\{3, r\}$. To find the exact value of r-dynamic chromatic number of $C_n \trianglerighteq P_2$, we define three cases, namely $\chi(C_n \trianglerighteq P_2)$, $\chi_2(C_n \trianglerighteq P_2)$ and $\chi_{r\geq 3}(C_n \trianglerighteq P_2)$.

Case 1. For $r \geq 2$, the lower bond $\Delta(G) \leq \chi(G) \leq \Delta(G) + 1$, that $\chi(C_n \trianglerighteq P_2) \geq 3$. Furthermore, to show that $\chi(C_n \trianglerighteq P_2) \leq 3$ with coloring edges $E(C_n \trianglerighteq P_2)$ as in function c_1. Let $D = \{1, 2, \ldots, k\}$ is set of color from k-coloring that c_1 the function by defining a map edges coloring D, $c_1 : E(C_n \trianglerighteq P_2) \to D$, so mapping each edges to set color D, by the following that:

$$\begin{cases} 12 \ldots 12, \\ n \text{ even} \\ 12 \ldots 12\ 3, \\ n \text{ odd} \end{cases}$$

$$c_1(x_1y_1, x_2y_2, \ldots, x_ny_n) = \begin{cases} 2\ 33 \ldots 33\ 1, \\ n \text{ odd} \\ 33 \ldots 33, \\ n \text{ even} \end{cases}$$

From function coloring c_1 seen that the chromatic number is $\chi(C_n \trianglerighteq P_2) \leq 3$. Because $\chi(C_n \trianglerighteq P_2) \leq 3$ and $\chi(C_n \trianglerighteq P_2) \geq 3$, then $\chi(C_n \trianglerighteq P_2) = 3$, so that $\chi(C_n \trianglerighteq P_2) = \chi_2(C_n \trianglerighteq P_2) = 3$.

Case 2. For $r =\geq 3$, the lower bond $\chi_3(G) \geq \chi_2(G)$, that $\chi_3(C_n \trianglerighteq P_2) \geq 4$. We will prove that $\chi_3(C_n \trianglerighteq P_2) = 3$ such as coloring function c_1 that the definition edges coloring r-dynamis not full filled. It is caused by edge x_1x_2, earned $d(u) + d(v) - 2 = 4$, $|c(N(e))| = 2$ dan $\min\{r, d(u) + d(v) - 2\} = \min\{3, 4\} = 3$, so that $2 \not\geq 3$. So, lower bound is $\chi_3(C_n \trianglerighteq P_2) \geq 4$. Furthermore, to show that $\chi_3(C_n \trianglerighteq P_2) \leq 4$ with coloring edge $E(C_n \trianglerighteq P_2)$ as in function c_2. Let $D = \{1, 2, \ldots, k\}$ is set of color from k-coloring that c_2 the function by defining a map edges coloring D, $c_2 : E(C_n \trianglerighteq P_2) \to D$, so mapping each edges to set color D, by the following that:

$$c_2(x_1x_2, x_2x_3, \ldots, x_nx_1) = \begin{cases} 12 \ldots 12, \\ n \text{ even} \\ 12 \ldots 12\ 3, \\ n \text{ odd} \end{cases}$$

$$c_2(x_1y_1, x_2y_2, \ldots, x_ny_n) = \begin{cases} 43 \ldots 43\ 4, \\ n \text{ odd} \\ 34 \ldots 34, \\ n \text{ even} \end{cases}$$

From function coloring c_2 seen that the chromatic number is $\chi(C_n \trianglerighteq P_2) \leq 4$. Because $\chi(C_n \trianglerighteq P_2) \leq 4$ and $\chi(C_n \trianglerighteq P_2) \geq 4$, then $\chi(C_n^{P_2}) = 4$, so that $\chi(C_n \trianglerighteq P_2) = \chi_{r\geq 3}(C_n \trianglerighteq P_2) = 4$. □

Theorem 3. *Let G be a comb product denote by $C_n \trianglerighteq C_m$ for $n \geq 3$ and $m \geq 3$, vertex r-dynamic chromatic number of $C_n \trianglerighteq C_m$ is:*

$$\chi(C_n \trianglerighteq C_m) = \begin{matrix} 2,\ n \text{ even} \\ 3,\ n \text{ otherwise} \end{matrix}$$

$$\chi_2(C_n \trianglerighteq C_m) = \begin{cases} 3,\ n = 3,\ m = 3 \end{cases}$$

$$\chi_3(C_n \trianglerighteq C_m) = 4,\ n = 3,\ m = 3$$

$$\chi_{r\geq 2}(C_n \trianglerighteq C_m) = \begin{cases} 4,\ \ n = 5,\ m = 5 \\ 5,\ \ n = 3k,\ m = 3k \end{cases}$$

$$\chi_{r\geq 4}(C_n \trianglerighteq C_m) = \begin{cases} 5, \quad n = 3k,\ i \leq i \leq n, \\ \quad 1 \leq j \leq m-2 \end{cases}$$

Proof. The comb product of graph denoted by $C_n \trianglerighteq C_m$, is connected graph with vertex set $V(C_n \trianglerighteq C_m) = \{x_i, ; 1 \leq i \leq n\} \cup \{x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m-2\}$ and edge set $E(C_n \trianglerighteq C_m) = \{x_nx_1; x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{x_{i,j}x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m-3\} \cup \{x_ix_{i,1}; 1 \leq i \leq n\} \cup \{x_1x_{n,m-2}; x_{i+1}x_{i+1,1}; n \leq i \leq n\}$. Thus, the order and size of this graph are $p = |V(C_n \trianglerighteq C_m)| = n + n(m-2), q = |E(C_n \trianglerighteq C_m)| = mn$. Since all edges in C_n joint with one edge in C_m, it gives $\Delta(C_n \trianglerighteq C_m) = 4$.

By Observation 1, $\chi(C_n \trianglerighteq C_m) \geq \min\{r, \Delta(C_n \trianglerighteq C_m)\} = \min\{r, 4\}$. To find the vertex r-dynamic chromatic number of $(C_n \trianglerighteq C_m)$, we define three cases, namely for $\chi(C_n \trianglerighteq C_m), \chi_d(C_n \trianglerighteq C_m), \chi_3(C_n \trianglerighteq C_m)$ and $\chi_4(C_n \trianglerighteq C_m)$.

For $\chi(C_n \trianglerighteq C_m), \chi_d(C_n \trianglerighteq C_m)$, the lower bound $\chi_1(C_n \trianglerighteq C_m) \geq \min\{2, 4\} = 2$. We will show that $\chi_1(C_n \trianglerighteq C_m) \leq 3$, by defining a map $c_4 : V(C_n \trianglerighteq C_m) \to \{1, 2, 3, \ldots, k\}$ where $n \geq 3$, $m \geq 3$ by the following :

- For n and m even,

$$c_4(x_i) = \begin{cases} 1, & i \equiv 1(mod\, 2),\ 1 \leq i \leq n, \\ 2, & i \equiv 0(mod\, 2),\ 1 \leq i \leq n. \end{cases}$$

$$c_4(x_{i,j}) = \begin{cases} 2, & i \equiv 1(mod\, 2),\ j \equiv 1 \\ & (mod\, 2),\ 1 \leq i \leq n, \\ & 1 \leq j \leq m-2, \\ 1, & i \equiv 1(mod\, 2),\ j \equiv 0 \\ & (mod\, 2),\ 1 \leq i \leq n, \\ & 2 \leq j \leq m-2, \\ 2, & i \equiv 2(mod\, 2),\ j \equiv 2 \\ & (mod\, 2),\ 1 \leq i \leq n, \\ & 1 \leq j \leq m-2, \\ 1, & i \equiv 0(mod\, 2),\ j \equiv 1 \\ & (mod\, 2),\ 2 \leq i \leq n, \\ & 1 \leq j \leq m-2, \end{cases}$$

- For n and m odd,

$$c_4(x_i)=\begin{cases}1, & i\equiv 1(mod\,2),\ 1\le i\le n-1,\\ 2, & i\equiv 0(mod\,2),\ 1\le i\le n-1,\\ 3, & i=n.\end{cases}$$

$$c_4(x_{i,j})=\begin{cases}1, & i\equiv 1(mod\,2),\ j\equiv 0\ (mod\,2),\ 1\le i\le n-1,\ 2\le j\le m-2,\\ 1, & i\equiv 0(mod\,2),\ j\equiv 1\ (mod\,2),\ 2\le i\le n,\ 1\le j\le m-3, x_{i-1}, x_{m-2}\\ 2, & i\equiv 1(mod\,2),\ j\equiv 1\ (mod\,2),\ 1\le i\le n,\ 1\le j\le m-3,\\ 2, & i\equiv 0(mod\,2),\ j\equiv 0(mod\,2),\ 2\le i\le n-1,\ 1\le j\le m-3,\\ 3, & x_n; x_{i,m-3};\ 1\le i\le n-3\end{cases}$$

- For $n=odd$ and $m=even$,

$$c_4(x_i)=\begin{cases}1, & i\equiv 1(mod\,2),\ 1\le i\le n-1,\\ 2, & i\equiv 0(mod\,2),\ 1\le i\le n-1,\\ 3, & i=n.\end{cases}$$

$$c_4(x_{i,j})=\begin{cases}1, & i\equiv 1(mod\,2),\ j\equiv 0\ (mod\,2),\ 1\le i\le n,\ 1\le j\le m-2,\\ 2, & i\equiv 0(mod\,2),\ j\equiv 1\ (mod\,2),\ 1\le i\le n.\end{cases}$$

- For $n=even$ and $m=odd$,

$$c_4(x_i)=\begin{cases}1, & i\equiv 1(mod\,2),\ 1\le i\le n,\\ 2, & i\equiv 0(mod\,2),\ 1\le i\le n.\end{cases}$$

$$c_4(x_{i,j})=\begin{cases}1, & i\equiv 1(mod\,2),\ j\equiv 0\ (mod\,2),\ 1\le i\le n,\ 2\le j\le m-3,\\ 1, & i\equiv 0(mod\,2),\ j\equiv 1\ (mod\,2),\ 1\le i\le n,\ 1\le j\le m-2,\\ 2, & i\equiv 1(mod\,2),\ j\equiv 1(mod\,2),\\ 2, & i\equiv 0(mod\,2),\ j\equiv 0(mod\,2),\\ 3, & 1\le i\le n; j=m-2\end{cases}$$

- For $\chi_{r=2}$, $n=3$ and $m=3,6$,

$$c_4(x_i)=i; 1\le i\le 3$$

$$c_4(x_{i,j})=\begin{cases}3, & i=1, j=1\\ 1, & i=2, j=1\\ 2, & i=3, j=1\end{cases}$$

It easy to see that c_4 gives $\chi(C_n\trianglerighteq C_m)\le 3$ and $\chi_d(C_n\trianglerighteq C_m)\le 3$. Thus $\chi(C_n\trianglerighteq C_m)=3$ and $\chi_d(C_n\trianglerighteq C_m)=3$.

For $r=3$, the lower bound $\chi_3(C_n\trianglerighteq C_m)\ge \min\{3,4\}=3$. We will show that $\chi_3(C_n\trianglerighteq C_m)\le 4$, by defining a map $c_5: V(C_n\trianglerighteq C_m)\to\{1,2,3,\dots,k\}$ where $n\ge 3$ and $m\ge 3$ by the following:

- For $n=3$ and $m=3,6$

$$c_5(x_i)=i$$

$$c_5(x_{i,j})=\begin{cases}321\,4; i=1,\\ 132\,4; i=2,\\ 213\,4; i=3,\end{cases}$$

$$c_5(x_i)=i, c_5(x_{i,j})=4; n=3, m=3.$$

- For $n=3k$ and $m=3k$

$$c_5(x_i)=\begin{cases}1, & i\equiv 1(mod\,3),\\ 2, & i\equiv 2(mod\,3),\\ 3, & i\equiv 3(mod\,3),\end{cases}$$

$$c_5(x_{i,j})=\begin{cases}321\,321\cdots 321\,4, & i\equiv 1\ (mod\,3),\\ 132\,132\cdots 132\,4, & i\equiv 2\ (mod\,3),\\ 213\,213\cdots 213\,4, & i\equiv 3\ (mod\,3),\end{cases}$$

It is easy to that c_5 gives $\chi_3(C_n\trianglerighteq C_m)\le 4$. Thus $\chi_3(C_n\trianglerighteq C_m)=4$.

For $r=4$, the lower bound $\chi_4(C_n\trianglerighteq C_m)\ge \min\{4,4\}=4$. We will show that $\chi_4(C_n\trianglerighteq C_m)\le 5$, by defining a map $c_6: V(C_n\trianglerighteq C_m)\to\{1,2,3,\dots,k\}$ where $n\ge 3$ and $m\ge 3$ by the following:

- For $n=3k$ and $m=3k$, $1\le i\le n, 1\le j\le m-2$

$$c_6(x_i)=\begin{cases}1, & i\equiv 1(mod\,3),\\ 2, & i\equiv 2(mod\,3),\\ 3, & i\equiv 3(mod\,3).\end{cases}$$

$$c_6(x_{i,j})=\begin{cases}421\,421\cdots 421\,5, & i\equiv 1\ (mod\,3),\\ 432\,432\cdots 432\,5, & i\equiv 2\ (mod\,3),\\ 413\,413\cdots 413\,5, & i\equiv 3\ (mod\,3),\end{cases}$$

- For $n=3k$ and $m=3k+1, 1\le i\le n, 1\le j\le m-2$

$$c_6(x_i)=\begin{cases}1, & i\equiv 1(mod\,3),\\ 2, & i\equiv 2(mod\,3),\\ 3, & i\equiv 3(mod\,3).\end{cases}$$

$$c_6(x_{i,j})=\begin{cases}421\,421\cdots 421\,45, & i\equiv 1\ (mod\,3),\\ 432\,432\cdots 432\,45, & i\equiv 2\ (mod\,3),\\ 415\,415\cdots 415\,45, & i\equiv 3\ (mod\,3),\end{cases}$$

It is easy to that c_6 gives $\chi_4(C_n\trianglerighteq C_m)\le 5$. Thus $\chi_4(C_n\trianglerighteq C_m)=5$.

For $r=5$, the lower bound $\chi_5(C_n\trianglerighteq C_m)\ge \min\{5,4\}=4$. We will show that $\chi_5(C_n\trianglerighteq C_m)\le 5$, by defining a map $c_7: V(C_n\trianglerighteq C_m)\to\{1,2,3,\dots,k\}$ where $n\ge 2$ by the following:

- For $n=5$ and $m=3k$

$$c_7(x_i)=i$$

$$c_7(x_{i,j})=\begin{cases}421\,421\cdots 421,\ 5 & i=1,\\ 432\,432\cdots 432,\ 1 & i=2,\\ 543\,543\cdots 543,\ 1 & i=3,\\ 254\,254\cdots 254,\ 3 & i=4,\\ 215\,215\cdots 215,\ 3 & i=5,\end{cases}$$

- For $n = 5$ and $m = 3k + 1$

$$c_7(x_i) = i$$

$$c_7(x_{i,j}) = \begin{cases} 421\,421 \cdots 421,\ 5 & i = 1, \\ 324\,324 \cdots 324,\ 1 & i = 2, \\ 543\,543 \cdots 543,\ 1 & i = 3, \\ 254\,254 \cdots 254,\ 3 & i = 4, \\ 215\,215 \cdots 215,\ 3 & i = 5, \end{cases}$$

It is easy that c_7 gives $\chi_5(C_n \trianglerighteq C_m) \leq 5$. Thus $\chi_5(C_n \trianglerighteq C_m) = 5$. Since for $r \geq 5$, we have $r \geq \Delta(C_n \trianglerighteq C_m)$. By Observation 1, $\chi_r(C_n \trianglerighteq C_m) = \chi_5(C_n \trianglerighteq C_m) = 5$. It concludes the proof.

□

Theorem 4. *Let G be a comb product denote by $C_n \trianglerighteq C_m$ for $n \geq 3$ and $m \geq 3$, edges r-dynamic chromatic number of ($C_n \trianglerighteq C_m$) is :*

$$\chi_{1 \leq r \leq 3}(C_n \trianglerighteq C_m) = 4$$

Proof. The graph $(C_n \trianglerighteq C_m)$ is a connected graph with vertex set $V(C_n \trianglerighteq C_m)$ $= \{x_i, ; 1 \leq i \leq n\} \cup \{x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m-2\}$ and edge set $E(C_n \trianglerighteq C_m)$ $= \{x_n x_1; x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_{i,j} x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m-3\} \cup \{x_i x_{i,1}; 1 \leq i \leq n\} \cup \{x_1 x_{n,m-2}; x_{i+1} x_{i+1,1}; n \leq i \leq n\}$. Thus, the order and size of this graph are $p = |V(C_n \trianglerighteq C_m)| = n + n(m-2), q = |E(C_n \trianglerighteq C_m)| = mn$. Since all edges in C_n joint with one edge in C_m, it gives $\Delta(C_n \trianglerighteq C_m) = 4$.

By Observation 1, $\chi_r(C_n \trianglerighteq C_m) \geq \min\{\chi(C_n \trianglerighteq C_m), r\} + 1 = \min\{r, d(u) + d(v) - 2\}$. To find the exact value of r-dynamic chromatic number of $(C_n \trianglerighteq C_m)$, we define three cases, namely for $\chi_{1 \leq r \leq 4}(C_n \trianglerighteq C_m)$, $\chi_5(C_n \trianglerighteq C_m)$.

For $r = 1$, the lower bound $\Delta(G) \leq \chi(G) \leq \Delta(G) + 1$, that $\chi(C_n \trianglerighteq C_m) \geq 3$. Furthermore, to show that $\chi(C_n \trianglerighteq C_m) \leq 3$ with coloring edges $E(C_n \trianglerighteq C_m)$ as in function c_8. Let $D = \{1, 2, \ldots, k\}$ is set of color from k-coloring that c_1 the function by defining a map edges coloring D, $c_8 : E(C_n \trianglerighteq C_m) \rightarrow D$, so mapping each edges to set color D, by the following that:

$$c_8(x_i x_{i+1}) = \begin{cases} 1, \ i \text{ odd } 1 \leq i \leq n-1 \\ 2, \ i \text{ even } 1 \leq i \leq n-1 \end{cases}$$

$$c_8(x_n x_{i+1}) = \begin{cases} 2, \ i \text{ odd } 1 \leq i \leq n-1 \\ 3, \ i \text{ even } 1 \leq i \leq n-1 \end{cases}$$

$$c_8(x_i y_{1,j}) = 4, \ 1 \leq i \leq n \ 1 \leq j \leq m$$

- For $n = even$

$$c_8(y_{i,j} y_{i,j+1}) = \begin{cases} 1, & \text{j} \equiv 1 (\bmod 3), \\ & 1 \leq i \leq m-3 \\ 2, & \text{j} \equiv 2 (\bmod 3), \\ & 1 \leq i \leq m-3 \\ 4, & \text{j} \equiv 0 (\bmod 3), \\ & 1 \leq i \leq m-3 \end{cases}$$

$$c_8(x_{i+1} y_{i,m-2}) = 3$$

$$c_8(x_1 y_{n,m-2}) = 3$$

- For $n = even$

$$c_8(y_{i,j} y_{i,j+1}) = \begin{cases} 1, & \text{j} \equiv 1 (\bmod 3), \ i = n \\ & 1 \leq i \leq n-2; \\ & 1 \leq i \leq m, \\ 2, & \text{j} \equiv 2 (\bmod 3), \\ & i = n-1 \\ & 2 \leq i \leq n-2; \\ & 2 \leq i \leq m, \\ 3, & \text{j} \equiv 2 (\bmod 3), \\ & n-1 \leq i \leq n \\ 4, & 1 \leq i \leq n; \\ & 1 \leq j \leq m \end{cases}$$

$$c_8(x_{i+1} y_{i,m-2}) = 3, \ 1 \leq i \leq n-2$$

$$c_8(x_1 y_{n,m-2}) = 2$$

$$c_8(x_{n-1} y_{n-1,m-2}) = 1$$

From function coloring c_8 seen that the chromatic number is $\chi(C_n \trianglerighteq C_m) \leq 4$. Because $\chi(C_n \trianglerighteq C_m) \leq 4$ and $\chi(C_n \trianglerighteq C_m) \geq 4$, then $\chi(C_n \trianglerighteq C_m) = 4$, so that $\chi(C_n \trianglerighteq C_m) = \chi_2(C_n \trianglerighteq C_m) = \chi_3(C_n \trianglerighteq C_m) = 4$.

CONCLUSIONS

We have found some edge and vertex $r-$dynamic chromatic number of several graphs, namely comb product of graph $C_n \trianglerighteq P_2$ and $C_n \trianglerighteq C_m$. It is interesting to characterize a property of any graph operation to have an exact value or upper bound of their $r-$dynamic chromatic numbers.

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REFERENCES

[1] B. Montgomery, "Dynamic Coloring of Graphs", Ph.D Dissertation, West Virginia University, 2001.

[2] S. Akbari, M. Ghanbari and S. Jahanbekam, "On the dynamic chromatic number of graphs", in: Combinatorics and Graphs, in: Contemp. Math., Amer. Math. Soc., vol. 531, pp. 1-18, 2010.

[3] H.J. Lai, B. Montgomery and H. Poon, "Upper bounds of dynamic chromatic number", Ars Combin., vol. 68, pp. 193–201, 2003.

[4] S. Jahanbekama, J. Kimb, O. Suil and D. B. Westd, "On r-dynamic Coloring of Graphs", Discrete Applied Mathematics, 2016, In Press.

[5] M. Alishahi, "Dynamic chromatic number of regular graphs", Discrete Appl. Math., vol. 160, pp. 2098-2103, 2012.

[6] M. Alishahi, "On the dynamic coloring of graphs", Discrete Appl. Math., vol. 159, pp. 152-156, 2011.

[7] S. Akbari, M. Ghanbari and S. Jahanbekam, "On the list dynamic coloring of graphs", Discrete Appl. Math., vol. 157, pp. 3005-3007, 2009.

[8] A. Taherkhani, "r-Dynamic Chromatic Number of Graphs", arXiv:1401.6470v1 [math.CO], Iran: Dept of Math, Institutu for Basic Studies and Advance Sciense, 2014.

[9] D. Meganingtyas, Dafik and Slamin, "On r-dynamic Coloring of Operation Product of Cycle and Path Graph", University of Jember, 2016.