On Total r-Dynamic Coloring of Several Classes of Graphs and Their Related Operations

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Abstract—All graphs in this paper are simple, connected and undirected. Let r, k be natural numbers. By a proper k-coloring of a graph G, we mean a map $c : V(G) \to S$, where $|S| = k$, such that any two adjacent vertices receive different colors. A total r-dynamic coloring is a proper k-coloring c of G, such that $\forall v \in V(G)$, $|c(N(v))| \geq min[r, d(v) + |N(v)|]$ and $\forall uv \in E(G)$, $|c(N(uv))| \geq min[r, d(u) + d(v)]$. The total r-dynamic chromatic number, written as $\chi''_r(G)$, is the minimum k such that G has an r-dynamic k-coloring. Finding the total r-dynamic chromatic number is considered to be a NP-Hard problems for any graph. Thus, in this paper, we initiate to study $\chi''_r(G)$ of several classes of graphs and and their related operations.

Keywords—Total r-Dynamic Chromatic Number, Several Slasses of Graphs, Graph Operations.

INTRODUCTION

Graph G is a couple of $(V(G), E(G))$ with $V(G)$ is finite set not empty of elements called vertex, and $E(G)$ is a set (maybe empty) of a pair not ordered (u, v) called edge [1]. A graph G possible not having edge, but must be having vertex at least one. A graph who do not have edges but having a vertex only called by trivial graph. [2]. The number of vertices on a graph called vertex cardinality and denoted by $|V|$ while the number of edges on a graph called edge cardinality with denoted by $|E|$ [3].

One study in graph theory is graph coloring. Graph coloring be a function that maps elements elements to any set of. If the domain set was a edge called edge coloring. If the domain set was vertex hence called vertex coloring. If the domain set vertex and edge called total coloring. Total coloring is a function c that maps $(V(G), E(G))$ to the set of color so much for any two vertices neighbors, and every two edges neighbors and any vertex of which is one side with random edge has of different colors. Minimum number of colors called to the chromatic number, and always based alleged 1 as follows:

Conjecture 1 According to behzad and vizing the chromatic number of total for each graph G must satisfy $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$ [4]

One study in graph theory is a total r-dynamic coloring developed from the vertex and edge r-dynamic coloring. Total coloring k-color r-dynamic is total coloring for every $v \in V(G)$ so $|c(N(v))| \geq min[r, d(v) + |N(v)|]$, and every edge $e = uv \in E(G)$ so much $|c(N(e))| \geq min[r, d(v) + d(u)]$ where $N(v)$ is neighbor v dan $c(N(v))$ is color that used by vertex neighbors v and $N(e)$ is neighbors edge e and $c(N(e))$ is color that used by edge neighbors edge e. Minimum number k so graph G satisfy total coloring k-color r-dynamic called chromatic number total r-dynamic denoted by $\chi''(G)$. The following are several definition operation graph used in this article.

Definition 1. *Shackle graph H denoted by $G = shack(H, v, n)$ is graph G generate of non trivial graph $H_1, H_2, ..., H_n$ so for all $1 \leq s, t \leq n$, H_s and H_t not having a vertex liaison where $|s - t| \geq 2$ and for all $1 \leq i \leq n-1$, H_i and H_{i+1} have exactly one vertex fellowship v, called with a vertex liaison and $k - 1$ connecting the vertex was different. If $G = shack(H, v, n)$ liaison vertex replaced by subgraph $K \subset H$ called by generalized shackle, dan denoted by $G = gshack(H, K \subset H, n)$ [5].*

THE RESULTS

The results from the study is the definition and a new theorem related a total r-dynamic coloring. Definition of total r-dynamic coloring can be seen in definition 2.

The theorem about a total r-dynamic coloring on graph of path, *shackle* book graph $(shack(B_2, v, n)$ dan graph operation *generalized shackle* graph *friendship* $gshack(\mathbf{F}_4, e, n)$.

Definition 2. *Let $D = \{1, 2, 3, \ldots, k\}$ is set of color by k colors dan c is function maps all vertices and edges G to set of colors. Total r-dynamic coloring of graph G is defined as mapping c from $(V(G) \cup E(G))$ to D so satisfy a condition the following :*

1. $\forall v \in V(G)$, $|c(N(v))| \geq min[r, d(v) + |N(v)|]$ *dan*
2. $\forall e = uv \in E(G)$, $|c(N(e))| \geq min[r, d(v) + d(u)]$

Observation 1. *Let $\Delta(G)$ is maximum degree of graph G so $\chi''(G) \leq \chi''_d(G) \leq \chi''_3(G) \leq \cdots \leq \chi''_{\Delta(G)}(G)$.*

$\diamondsuit$ **Theorem 1.** *Let G is a path P_n. For $n \geq 3$, chromatic number total r-dynamic of graph G is*

$$\chi''_r(P_n) = \begin{cases} 3, \text{ for } 1 \leq r \leq 2 \\ 4, \text{ for } r = 3 \\ 5, \text{ for } r \geq 4 \end{cases}$$

Proof. Set of vertices and set of edges of path for $n \geq 3$ is $V(P_n) = \{x_i; 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1}; 1 \leq i \leq n-1\}$, so $|V(P_n)| = n$ and $|E(P_n)| = n-1$ and $\Delta(G) = 2$.

Case 1. Based on Conjecture 1 that $\Delta(G) + 1 \leq \chi''_r(G) \leq \Delta(G) + 2$, so $\chi''(P_n) \geq 3$. To prove that chromatic number of total $1, 2$-dynamic coloring of path (P_n) is 3, needs to be proven $\chi''(P_n) \geq 3$ and $\chi'' P_n) \leq 3$. Then indicated that the number of chromatic $\chi''(P_n) \leq 3$ to coloring function c_1. Let $D = \{1, 2, 3, \ldots, k\}$ is set of colors by k colors

$$c_1(x_1, x_2, \ldots, x_i) = \begin{cases} 132\ldots132, \\ 1 \leq i \leq n, \\ i \equiv 0 (\text{mod } 3) \\ 132\ldots132\ 1, \\ 1 \leq i \leq n, \\ i \equiv 1 (\text{mod } 3) \\ 132\ldots132\ 13, \\ 1 \leq i \leq n, \\ i \equiv 2 (\text{mod } 3) \end{cases}$$

$$c_1(x_1x_2,\dots x_ix_{i+1}) = \begin{cases} 213\dots 213, \\ 1 \le i \le n, \\ i \equiv 0(\text{mod } 3) \\ \\ 213\dots 213\ 2, \\ 1 \le i \le n, \\ i \equiv 1(\text{mod } 3) \\ \\ 213\dots 213\ 21, \\ 1 \le i \le n, \\ i \equiv 2(\text{mod } 3) \end{cases}$$

From coloring function of c_1 it can be seen that the number of chromatic total in graph the is $\chi''(P_n) \le 3$. Since $\chi''(P_n) \le 3$ and $\chi''(P_n) \ge 3$ than $\chi''(P_n) = 3$, for $n \ge 3$.

Case 2. Base of Observation 1 that $\chi''_3(G) \ge \chi''_2(G)$, then $\chi''_3(P_n) \ge \chi''_d(P_n) = 3$. Let $\chi''_3(P_n) = 3$ as on coloring function c_1, So does not meet the definition of the total r-dynamic coloring. Leading to the need for additional colors become 4-coloring, $\chi''_3(P_n) \ge 4$.

To proven chromatic number of total 3-dynamic coloring of path (P_n) is 4, needs to be proven $\chi''_3(P_n) \ge 4$ and $\chi''_3(P_n) \le 4$. Then indicated that the number of chromatic $\chi''_3(P_n) \le 4$ by coloring function c_2. Let $D = \{1, 2, 3, \dots, k\}$ is set of colors by k color and c_2 is function who pairs every vertic es and edges to set D, $c_2 : (V(P_n) \cup E(P_n)) \to D$. For $n \ge 3$, coloring function c_2 is as follows:

$$c_2(x_1, x_2, \dots, x_i) = \begin{cases} 13\dots 13, \\ 1 \le i \le n, \\ n \text{ even} \\ \\ 13\dots 13\ 1, \\ 1 \le i \le n, \\ n \text{ odd} \end{cases}$$

$$c_2(x_1x_2,\dots x_ix_{i+1}) = \begin{cases} 24\dots 24, \\ 1 \le i \le n, \\ n \text{ even} \\ \\ 24\dots 24\ 2, \\ 1 \le i \le n, \\ n \text{ odd} \end{cases}$$

From c_2 it can be seen that the number of chromatic total 3-dynamic is $\chi''_3(P_n) \le 4$. Since $\chi''_3(P_n) \le 4$ and $\chi''_3(P_n) \ge 4$ than $\chi''_3(P_n) = 4$ so $\chi''_3(P_n) = 4$ for $n \ge 3$. as illustration, served figure 2 that is the total 3-dynamic coloring of path (P_n).

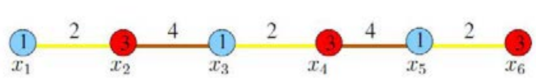

Fig 1. Total 3-Dynamic Coloring of Path (P_n)

Case 3. Based on Observation 1 that $\chi''_4(G) \ge \chi''_3(G)$, than $\chi''_4(P_n) \ge \chi''_3(P_n) = 4$. Let $\chi''_3(P_n) = 4$ as on coloring function c_2, so does no satisfy definition of total r-dynamic coloring. So that required the addition of colors become 5-coloring, $\chi''_4(P_n) \ge 5$.

To prove the chromatic number from the total 4-dynamic coloring of path (P_n) is 5, needs to be proven $\chi''_4(P_n) \ge 5$ and $\chi''_4(P_n) \le 5$. Then indicated that the chromatic number $\chi''_4(P_n) \le 5$ by coloring function c_3. Let $D = \{1, 2, 3, \dots, k\}$ is set of colors with k colors and c_3 is function pairing any vertex and edge to the set of color D, $c_3 : (V(P_n) \cup E(P_n)) \to D$. For $n \ge 3$, coloring function c_3 is as follows:

$$c_3(x_1, x_2, \dots, x_i) = \begin{cases} 13524\dots 13524, \\ 1 \le i \le n, \\ i \equiv 0(\text{mod } 3) \\ \\ 13524\dots 13524\ 1, \\ 1 \le i \le n, \\ i \equiv 1(\text{mod } 3) \\ \\ 13524\dots 13524\ 13, \\ 1 \le i \le n, \\ i \equiv 2(\text{mod } 3) \\ \\ 13524\dots 13524\ 135, \\ 1 \le i \le n, \\ i \equiv 3(\text{mod } 3) \\ \\ 13524\dots 13524\ 1352, \\ 1 \le i \le n, \\ i \equiv 4(\text{mod } 3) \end{cases}$$

$$c_3(x_1x_2, \dots, x_ix_{i+1}) = \begin{cases} 24135\dots \\ 24135, \\ 1 \le i \le n, \\ i \equiv 1(\text{mod } 3) \\ \\ 24135\dots \\ 24135\ 2, \\ 1 \le i \le n, \\ i \equiv 2(\text{mod } 3) \\ \\ 24135\dots \\ 24135\ 24, \\ 1 \le i \le n, \\ i \equiv 3(\text{mod } 3) \\ \\ 24135\dots \\ 24135\ 241, \\ 1 \le i \le n, \\ i \equiv 4(\text{mod } 3) \\ \\ 24135\dots \\ 24135\ 2413, \\ 1 \le i \le n, \\ i \equiv 0(\text{mod } 3) \end{cases}$$

Of coloring function on c_3 It is evident that the chromatic number of total 4-dynamic coloring is $\chi''_4(P_n) \le 5$. Since $\chi''_4(P_n) \le 5$ and $\chi''_4(P_n) \ge 5$. So can be concluded $\chi''_4(P_n) = 5$. So chromatic number total $\chi''_4(P_n) = 5$.

On path P_n, If reviewed from the vertices, number of $\min\{r, \max\{d(x_1) + |(N(x_i))|\}\} = \max\{d(x_1) + |(N(x_i))|\} = 4$. If in terms of coloring the edge of on path P_n, number of $\min\{r, \max\{d(u) + d(v)\}\} = \max\{d(u) + d(v)\} = 4$. Resulting in $\chi''_{r \ge 4}(P_n) = 5$. As illustration, served Figure 2 which is 4-dynamic coloring of path (P_n). Based on the description above, then Theorem 1 proved. □

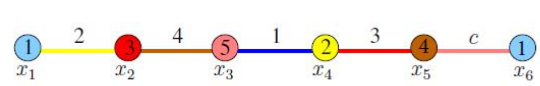

Fig 2. Total r-Dynamic Coloring of Path (P_n)

◊ **Theorem 2.** *Let graph G is operation graph shackle of book graph B_2. For $n \ge 2$, chromatic number of total r-dynamic shackle of book graph $Shack(B_2, v, n)$ is*

$$\chi''_r(Shack(B_2, v, n)) = \begin{cases} 5, \text{ for } 1 \le r \le 3 \\ 6, \text{ for } r = 4 \\ 9, \text{ for } r = 5 \\ 10, \text{ for } r \ge 6 \end{cases}$$

Proof. Set of vertices of graph $V(Shack(B_2, v, n)) = \{x_{ij}, z_{ij}; 1 \le i \le n, 1 \le j \le 2\} \cup \{y_i; 1 \le i \le n+1\}$ and set of edges $E(B_2, v, n)) = \{x_{ij}y_i; 1 \le i \le n, j = 1\} \cup \{x_{ij}y_{i+1}; 1 \le i \le n, j = 2\} \cup \{x_{ij}z_{ij+1}; 1 \le i \le n, j = 1\} \cup \{x_{ij+1}z_{ij}; 1 \le i \le n, j = 1\} \cup \{y_i z_{ij}; 1 \le i \le n, j = 1\} \cup \{y_{i+1}z_{ij}; 1 \le i \le n, j = 2\} \cup \{z_{ij}z_{ij+1}; 1 \le i \le n, j = 1\}$ so cardinality vertex and edges is $|V(Shack(B_2, v, n)| = 5n + 1$ and $|E(Shack(B_2, v, n))| = 7n$ and $\Delta(G) = 4$.

Case 1. Based on Conjecture 1 that $\Delta(G) + 1 \le \chi''_r(G) \le \Delta(G) + 2$, so $\chi''(Shack(B_2, v, n)) \ge 5$. To prove the chromatic number total 1,2,3-dynamic coloring of graph $(Shack(B_2, v, n))$ is 5, needs to be proven $\chi''(Shack(B_2, v, n)) \ge 5$ and $\chi''(Shack(B_2, v, n)) \le 5$. Then indicated that the chromatic number $\chi''(Shack(B_2, v, n)) \le 5$ by coloring function c_4. Let $D = \{1, 2, 3, \dots, k\}$ is the set of color with k colors and c_4 is function who pairing every vertex and edge to The set of color D, $c_4 : (V(Shack(B_2, v, n)) \cup E(Shack(B_2, v, n))) \to D$. Coloring function c_4 is as follows:

$$c_4(x_{ij}) = \begin{cases} 4, \ 1 \le i \le n, \ j = 1 \\ 5, \ 1 \le i \le n, \ j = 2 \end{cases}$$

$$c_4(z_{ij}) = \begin{cases} 3, \ 1 \le i \le n, \ j = 1 \\ 2, \ 1 \le i \le n, \ j = 2 \end{cases}$$

$$c_4(y_i) = 1, 1 \le i \le n + 1;$$
$$c_{23}(x_{ij}y_i) = 3, 1 \le i \le n, j = 1$$
$$c_4(x_{ij}y_{i+1}) = 3, 1 \le i \le n, j = 2;$$
$$c_4(y_i z_{ij}) = 5, 1 \le i \le n, j = 1$$
$$c_4(y_{i+1}z_{ij}) = 4, 1 \le i \le n, j = 2;$$
$$c_4(x_{ij}z_{ij+1}) = 2, 1 \le i \le n, j = 1$$
$$c_4(x_{ij+1}z_{ij}) = 3, 1 \le i \le n, j = 1;$$
$$c_4(z_{ij}z_{ij+1}) = 1, 1 \le i \le n, j = 1$$

Of coloring function on c_4 It is evident that the chromatic number Total of*shackle* book graph $(Shack(B_2, v, n))$ is $\chi''(Shack(B_2, v, n)) \le 5$. Since chromatic number $\chi''(Shack(B_2, v, n)) \le 5$ and $\chi''(Shack(B_2, v, n)) \ge 5$ So it can be concluded $\chi''(Shack(B_2, v, n)) = 5$. So that graph $G = Shack(B_2, v, n)$ have chromatic numbers $\chi''(G) = \chi''_d(G) = \chi''_3(G) = 5$.

Case 2. Based on Observation 1 that $\chi''_4(G) \ge \chi''_3(G)$, Can be concluded $\chi''_4(Shack(B_2, v, n)) \ge \chi''_3(Shack(B_2, v, n))$. Let $\chi''_4(Shack(B_2, v, n)) = 5$ As on coloring function c_4, So does not meet the definition total r-dynamic coloring. So that required the addition of color be 6-coloring, $\chi''_4(Shack(B_2, v, n)) \ge 6$.

To prove the chromatic number from the total 4-dynamic on graph $(Shack(B_2, v, n))$ is 6, need to be proven that $\chi''_4(Shack(B_2, v, n)) \ge 6$ and $\chi''_4(Shack(B_2, v, n)) \le 6$. Then indicated that the chromatic number $\chi''_4(Shack(B_2, v, n)) \le 6$ by coloring function c_5. Let $D = \{1, 2, 3, \dots, k\}$ is the set of color with k colors and c_5 is function who pairing every vertex and edges to set of colors D, $c_5 : V(Shack(B_2, v, n)) \cup E(Shack(B_2, v, n)) \to D$. Coloring function c_5 is as follows:

$$c_5(x_{ij}) = \begin{cases} 6, \ 1 \le i \le n, \ j = 1 \\ 3, \ 1 \le i \le n, \ j = 2 \end{cases}$$

$$c_5(z_{ij}) = \begin{cases} 4, \ 1 \le i \le n, \ j = 1 \\ 5, \ 1 \le i \le n, \ j = 2 \end{cases}$$

$$c_5(y_i) = 1, 1 \le i \le n + 1; \quad (1)$$
$$c_{24}(x_{ij}y_i) = 2, 1 \le i \le n, j = 1 \quad (2)$$
$$c_5(x_{ij}y_{i+1}) = 6, 1 \le i \le n, j = 2; \quad (3)$$
$$c_5(y_i z_{ij}) = 5, 1 \le i \le n, j = 1 \quad (4)$$
$$c_5(y_{i+1}z_{ij}) = 4, 1 \le i \le n, j = 2; \quad (5)$$
$$c_5(x_{ij}z_{ij+1}) = 3, 1 \le i \le n, j = 1 \quad (6)$$
$$c_5(x_{ij+1}z_{ij}) = 2, 1 \le i \le n, j = 1; \quad (7)$$
$$c_5(z_{ij}z_{ij+1}) = 6, 1 \le i \le n, j = 1 \quad (8)$$

Of coloring function on c_5 It is evident that the chromatic number total of *shackle* book graph, $G = (Shack(B_2, v, n))$ is $\chi''_4(G) \le 6$. Since $\chi''_4(G) \le 6$ dan $\chi''_4(G) \ge 6$ then $\chi''_4(G) = 6$ so chromatic number $\chi''_4(G) = 6$. As illustration, served Picture 4.24 Which is coloring 4-dynamic of *shackle* book graph B_2.

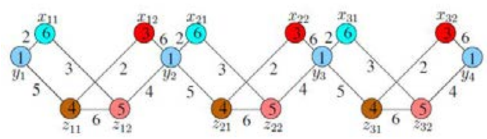

Fig 3. Total 4-dynamic Coloring of Book Graph $Shack(B_2, v, n)$

Case 3. Based on Observation1 that $\chi''_5(G) \ge \chi''_4(G)$, Can be concluded $\chi''_5(Shack(B_2, v, n)) \ge \chi''_5(Shack(B_2, v, n))$. Let $\chi''_5(Shack(B_2, v, n)) = 6$ As on coloring function c_5, So does not meet definition of total r-dynamic coloring so that required the addition of colors become 7-coloring, then $\chi''_5(Shack(B_2, v, n)) \ge 7$. But with 7 coloring still not meet Total 5-dynamic coloring So that plus to 8 coloring. For 8 coloring there are the number of edge that do not meet the definition of total r-dynamic coloring so plus to 9 coloring so $\chi''_5(Shack(B_2, v, n)) \ge 9$.

To prove chromatic numbers of the total 5-dynamic coloring on graph $Shack(B_2, v, n)$ is 9, needs to be proven $\chi''_5(Shack(B_2, v, n)) \ge 9$ and $\chi''_5(Shack(B_2, v, n)) \le 9$. Then indicated that the chromatic number $\chi''_5(Shack(B_2, v, n)) \le 9$ by coloring function c_6. Let $D = \{1, 2, 3, \dots, k\}$ is set of colors with k colors and c_6 is function who pairing every vertex and edges to set of color D, $c_6 : (V(Shack(B_2, v, n)) \cup E(Shack(B_2, v, n))) \to D$. For $n \ge 2$, coloring function c_6 Is as follows:

$$c_6(x_{ij}) = \begin{cases} 4, \ 1 \le i \le n, \ j = 1 \\ 3, \ 1 \le i \le n, \ j = 2 \end{cases}$$

$$c_6(z_{ij}) = \begin{cases} 6, \ 1 \le i \le n, \ j = 1 \\ 2, \ 1 \le i \le n, \ j = 2 \end{cases}$$

$c_6(y_i) = 1, 1 \le i \le n + 1; c_6(x_{ij}y_i) = 8, 1 \le i \le n, j = 1 c_6(x_{ij}y_{i+1}) = 9, 1 \le i \le n, j = 2;$ $c_6(y_i z_{ij}) = 2, 1 \le i \le n, j = 1 c_6(y_{i+1}z_{ij}) = 5, 1 \le i \le n, j = 2;$ $c_6(x_{ij}z_{ij+1}) = 3, 1 \le i \le n, j = 1 c_6(x_{ij+1}z_{ij}) = 4, 1 \le i \le n, j = 1;$ $c_6(z_{ij}z_{ij+1}) = 7, 1 \le i \le n, j = 1$

Of coloring function on c_6 It is evident that the chromatic number total of *shackle* book graph, $G = (Shack(B_2, v, n))$ is $\chi''_5(G) \le 9$. Since $\chi''_5(G) \le 9$ and $\chi''_5(G) \ge 9$ then $\chi''_5(G) = 9$ so chromatic number $\chi''_5(G) = 9$.

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Case 4. Based on Observation 1 that $\chi''_6(G) \geq \chi''_5(G)$, Can be concluded $\chi''_6(Shack(B_2, v, n)) \geq \chi''_5(Shack(B_2, v, n))$. Let $\chi''_5(Shack(B_2, v, n)) = 9$ As on coloring function c_6, So does not meet definition of total r-dynamic coloring so that required the addition of colors become 10-coloring, then $\chi''_6(Shack(B_2, v, n)) \geq 10$.

To prove chromatic numbers of the total 6-dynamic coloring on graph $Shack(B_2, v, n)$ is 10, needs to be proven $\chi''_6(Shack(B_2, v, n)) \geq 10$ and $\chi''_6(Shack(B_2, v, n)) \leq 10$. Then indicated that the chromatic number $\chi''_6(Shack(B_2, v, n)) \leq 10$ by coloring function c_7. Let $D = \{1, 2, 3, \ldots, k\}$ is set of colors with k colors and c_7 is function who pairing every vertex and edges to set of color D, $c_7 : (V(Shack(B_2, v, n)) \cup E(Shack(B_2, v, n))) \to D$. Coloring function c_7 is as follows:

$$c_7(x_{ij}) = \begin{cases} 4, & 1 \leq i \leq n,\ j = 1 \\ 3, & 1 \leq i \leq n,\ j = 2 \end{cases}$$

$$c_7(z_{ij}) = \begin{cases} 6, & 1 \leq i \leq n,\ j = 1 \\ 8, & 1 \leq i \leq n,\ j = 2 \end{cases}$$

$$c_7(y_i) = 1, 1 \leq i \leq n+1$$

$$c_7(x_{ij}y_i) = 9, 1 \leq i \leq n, j = 1$$

$$c_7(x_{ij}y_{i+1}) = 10, 1 \leq i \leq n, j = 2$$

$$c_7(y_i z_{ij}) = 2, 1 \leq i \leq n, j = 1$$

$$c_7(y_{i+1} z_{ij}) = 5, 1 \leq i \leq n, j = 2$$

$$c_7(x_{ij} z_{ij+1}) = 3, 1 \leq i \leq n, j = 1$$

$$c_7(x_{ij+1} z_{ij}) = 4, 1 \leq i \leq n, j = 1$$

$$c_7(z_{ij} z_{ij+1}) = 7, 1 \leq i \leq n, j = 1$$

Of coloring function on c_7 It is evident that the chromatic number total of *shackle* book graph, $G = (Shack(B_2, v, n))$ is $\chi''_6(G) \leq 10$. Since $\chi''_6(G) \leq 10$ and $\chi''_6(G) \geq 10$ then $\chi''_6(G) = 10$ so chromatic number $\chi''_5(G) = 10$.

On graph operating *shackle* book graph B_2, if in terms of coloring the vertex is, number of $\min\{r, \max\{d(v) + |(N(v))|\}\} = \max\{d(z_{ij}) + |(N(z_{ij}))|\} = 8$. In the the edge of on operation graph *shackle* book graph B_2, number of $\min\{r, \max\{d(u) + d(v)\}\} = \max\{d(u) + d(v)\} = 7$, resulting in $\chi''_{r \geq 6}(Shack(B_2, v, n)) = 10$. This is because when $r \geq 6$ number $\min\{r, \max\{d(v) + |(N(v))|\}\} = \max\{d(z_{ij}) + |(N(z_{ij}))|\} = 6$ and number $\min\{r, \max\{d(u) + d(v)\}\} = \max\{d(u) + d(v)\} = 7$. From the above description, so Theorem 2 proven. □

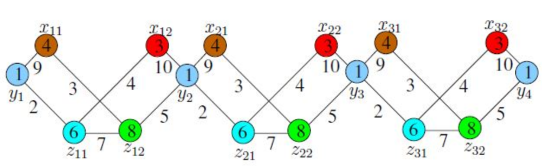

Fig 4. Total r-dynamic Coloring on Graph $Shack(B_2, v, n)$

◊ **Theorem 3.** *Let graph G is operation graph shackle of graph cocktail party $H_{2,2}$. For $n \geq 2$, chromatic number total r-dynamic graph $Shack(H_{2,2}, v, n)$ is*

$$\chi''_r(shack(H_{2,2}, v, n)) = \begin{cases} 5 \text{ untuk } 1 \leq r \leq 3 \\ 7, \text{ untuk } 4 \leq r \leq 5 \\ 10, \text{ untuk } 6 \leq r \leq 7 \\ 11, \text{ untuk } r \geq 8 \end{cases}$$

Proof. Set of vertices of graph $V(Shack(H_{2,2}, v, n)) = \{x_i; 1 \leq i \leq n+1\} \cup \{y_i, z_i; 1 \leq i \leq n\}$ and set of edges $E(H_{2,2}, v, n)) = \{x_i x_{i+1}; 1 \leq i \leq n\} \cup \{x_i z_i; 1 \leq i \leq n\} \cup \{y_i x_{i+1}; 1 \leq i \leq n\} \cup \{y_i z_i; 1 \leq i \leq n\}$. So cardinality vertex and edges is $|V(Shack(H_{2,2}, v, n)| = 3n + 1$ and $|E(Shack(H_{2,2}, v, n))| = 4n$, and $\Delta(G) = 4$.

Case 1. Based on Conjecture 1 that $\Delta(G) + 1 \leq \chi''_r(G) \leq \Delta(G) + 2$, so $\chi''(Shack(H_{2,2}, v, n)) \geq 5$. To prove the chromatic number total 1,2,3-dynamic coloring of graph $(Shack(H_{2,2}, v, n))$ is 5, needs to be proven $\chi''(Shack(H_{2,2}, v, n)) \geq 5$ and $\chi''(Shack(H_{2,2}, v, n)) \leq 5$. Then indicated that the chromatic number $\chi''(Shack(H_{2,2}, v, n)) \leq 5$ by coloring function c_{10}. Let $D = \{1, 2, 3, \ldots, k\}$ is the set of color with k colors and c_{10} is function who pairing every vertices and edges to the set of color D, $c_{19} : (V(Shack(H_{2,2}, v, n)) \cup E(Shack(H_{2,2}, v, n))) \to D$. Coloring function c_{10} is as follows:

$$c_{10}(x_i) = \begin{cases} 1, & 1 \leq i \leq n+1, \\ & i \equiv 1 (\text{mod } 3) \\ 3, & 1 \leq i \leq n+1, \\ & i \equiv 2 (\text{mod } 3) \\ 2, & 1 \leq i \leq n+1, \\ & i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{10}(y_i) = \begin{cases} 1, & 1 \leq i \leq n, \\ & i \equiv 1 (\text{mod } 3) \\ 3, & 1 \leq i \leq n, \\ & i \equiv 2 (\text{mod } 3) \\ 2, & 1 \leq i \leq n, \\ & i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{10}(z_i) = \begin{cases} 3, & 1 \leq i \leq n, \\ & i \equiv 1 (\text{mod } 3) \\ 2, & 1 \leq i \leq n, \\ & i \equiv 2 (\text{mod } 3) \\ 1, & 1 \leq i \leq n, \\ & i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{10}(x_i z_i) = \begin{cases} 2, & 1 \leq i \leq n, \\ & i \equiv 1 (\text{mod } 3) \\ 1, & 1 \leq i \leq n, \\ & i \equiv 2 (\text{mod } 3) \\ 3, & 1 \leq i \leq n, \\ & i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{10}(y_i x_{i+1}) = \begin{cases} 2, & 1 \leq i \leq n, \\ & i \equiv 1 (\text{mod } 3) \\ 1, & 1 \leq i \leq n, \\ & i \equiv 2 (\text{mod } 3) \\ 3, & 1 \leq i \leq n, \\ & i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{10}(x_i x_{i+1}) = \begin{cases} 4, & 1 \leq i \leq n,\ i \text{ odd} \\ 5, & 1 \leq i \leq n,\ i \text{ even} \end{cases}$$

$$c_{10}(y_i z_i) = \begin{cases} 4, & 1 \leq i \leq n,\ i \text{ odd} \end{cases}$$

$$5,\ 1 \leq i \leq n,\ i \text{ even}$$

Of coloring function on c_{10} tt is evident that the chromatic number total *shackle* graph *cocktail party* $H_{2,2}$, $G = Shack(H_{2,2}, v, n)$ is $\chi''(G) \leq 5$. Since $\chi''(G) \leq 5$

illustration, served Figure 7 who is 1,2,3-dynamic coloring *shackle* graph *cocktail party* $H_{2,2}$.

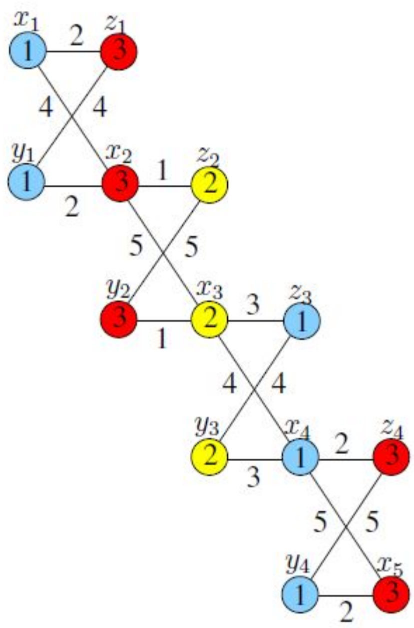

Fig 5. Total 1,2,3-dynamic Coloring on Graph $Shack(H_{2,2}, v, n)$

Case 2. Based on Observation 1 that $\chi''_4(G) \geq \chi''_3(G)$, Can be concluded $\chi''_4(Shack(H_{2,2}, v, n)) \geq \chi''_3(Shack(H_{2,2}, v, n))$.Let $\chi''_4(Shack(H_{2,2}, v, n)) = 5$ As on coloring function c_{10}, So does not meet the definition total r-dynamic coloring. So that required the addition of color become 6-coloring, $\chi''_4(Shack(H_{2,2}, v, n)) \geq 6$. But with 6-coloring still not meet total r-dynamic coloring, so that plus to 7 coloring, then $\chi''_4(Shack(H_{2,2}, v, n)) \geq 7$.

To prove chromatic numbers of the total 4-dynamic coloring on graph $Shack(H_{2,2}, v, n)$ is 7, needs to be proven $\chi''_4(Shack(H_{2,2}, v, n)) \geq 7$ and $\chi''_4(Shack(H_{2,2}, v, n)) \leq 7$. Then indicated that the chromatic number $\chi''_4(Shack(H_{2,2}, v, n)) \leq 7$ by coloring function c_{11}. Let $D = \{1, 2, 3, \ldots, k\}$ is set of colors with k colors and c_{11} is function who pairing every vertex and edges to set of color D, $c_{11} : (V(Shack(H_{2,2}, v, n)) \cup E(Shack(S_5, v, n))) \to D$. Coloring function c_{11} is as follows:

$$c_{11}(z_i) = \begin{cases} 3, & 1 \leq i \leq n, \\ & i \equiv 1 (\text{mod } 3) \\ 2, & 1 \leq i \leq n, \\ & i \equiv 2 (\text{mod } 3) \\ 5, & 1 \leq i \leq n, \\ & i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{11}(x_i) = \begin{cases} 1, & i = 1 \\ 5, & 2 \leq i \leq n+1, \\ & i \equiv 2 (\text{mod } 3) \\ 3, & 2 \leq i \leq n+1, \\ & i \equiv 0 (\text{mod } 3) \\ 2, & 2 \leq i \leq n+1, \\ & i \equiv 1 (\text{mod } 3) \end{cases}$$

$$c_{11}(x_i z_i) = \begin{cases} 2, & i = 1 \\ 3, & 2 \leq i \leq n, \\ & i \equiv 2 (\text{mod } 3) \\ 2, & 2 \leq i \leq n, \\ & i \equiv 0 (\text{mod } 3) \\ 5, & 2 \leq i \leq n, \end{cases}$$

$$c_{11}(y_i) = \begin{cases} 4, & 1 \leq i \leq n, \ i \text{ odd} \\ 1, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{11}(x_i x_{i+1}) = \begin{cases} 6, & 1 \leq i \leq n, \ i \text{ odd} \\ 7, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{11}(y_i z_i) = \begin{cases} 7, & 1 \leq i \leq n, \ i \text{ odd} \\ 6, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{11}(y_i x_{i+1}) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ odd} \\ 4, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

Of coloring function on c_{11} It is evident that the chromatic number total *shackle* graph *cocktail party* $H_{2,2}$ is $\chi''(Shack(H_{2,2}, v, n)) \leq 7$. Since chromatic number $\chi''(Shack(H_{2,2}, v, n)) \leq 7$ and $\chi''(Shack(H_{2,2}, v, n)) \geq 7$ Can be concluded $\chi''(Shack(H_{2,2}, v, n)) = 7$ so $\chi''_4(Shack(H_{2,2}, v, n)) = \chi''_5(Shack(H_{2,2}, v, n)) = 7$.

Case 3. Based on Observation 1 that $\chi''_6(G) \geq \chi''_5(G)$, can be concluded $\chi''_6(Shack(H_{2,2}, v, n)) \geq \chi''_5(Shack(H_{2,2}, v, n))$. Let $\chi''_6(Shack(H_{2,2}, v, n)) = 7$ as on coloring function c_{11}, so does not meet definition of total r-dynamic coloring. so that required the addition of colors become 8-coloring, $\chi''_6(Shack(H_{2,2}, v, n)) \geq 8$. But with 78 coloring still not meet total r-dynamic coloring, so that plus to 9 coloring, then chromatic number $\chi''_6(Shack(H_{2,2}, v, n)) \geq 9$.

To prove chromatic numbers of the total 6-dynamic coloring on graph $Shack(H_{2,2}, v, n)$ is 9, needs to be proven $\chi''_6(Shack(H_{2,2}, v, n)) \geq 9$ and $\chi''_6(Shack(H_{2,2}, v, n)) \leq 9$. Then indicated that the chromatic number $\chi''_6(Shack(H_{2,2}, v, n)) \leq 9$ by coloring function c_{12}. Let $D = \{1, 2, 3, \ldots, k\}$ is set of colors with k colors and c_{12} is function who pairing every vertex and edges to set of color D, $c_{12} : (V(Shack(H_{2,2}, v, n)) \cup E(Shack(H_{2,2}, v, n))) \to D$. Coloring function c_{12} is as follows:

$$c_{12}(x_i) = \begin{cases} 1, & 1 \leq i \leq n+1, \ i \text{ odd} \\ 6, & 1 \leq i \leq n+1, \ i \text{ even} \end{cases}$$

$$c_{12}(y_i) = \begin{cases} 4, & 1 \leq i \leq n, \ i \text{ odd} \\ 5, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{12}(z_i) = \begin{cases} 3, & 1 \leq i \leq n, \ i \text{ odd} \\ 2, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{12}(x_i z_i) = \begin{cases} 2, & 1 \leq i \leq n, \ i \text{ odd} \\ 3, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{12}(x_i x_{i+1}) = \begin{cases} 7, & 1 \leq i \leq n, \ i \text{ odd} \\ 9, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{12}(y_i x_{i+1}) = \begin{cases} 5, & 1 \leq i \leq n, \ i \text{ odd} \\ 4, & 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{12}(y_i z_i) = 8, 1 \leq i \leq n$$

Of coloring function on c_{12} It is evident that the chromatic number total *shackle* graph *cocktail party* $H_{2,2}$ is $\chi''_6(Shack(H_{2,2}, v, n)) \leq 9$. Since chromatic number $\chi''_6(Shack(H_{2,2}, v, n)) \leq 9$ and $\chi''_6(Shack(H_{2,2}, v, n)) \geq 9$ can be concluded $\chi''_6(Shack(H_{2,2}, v, n)) = 9$ so $\chi''_6(Shack(H_{2,2}, v, n)) =$

$\chi''_7(Shack(H_{2,2}, v, n)) = 9$.

Case 4. Based on Observation 1 that $\chi''_8(G) \geq \chi''_7(G)$, can be concluded $\chi''_8(Shack(H_{2,2}, v, n)) \geq \chi''_7(Shack(H_{2,2}, v, n))$. Let $\chi''_8(Shack(H_{2,2}, v, n)) = 9$ as on coloring function c_{12}, So does not meet definition of total r-dynamic coloring. So that required the addition of colors become 10-coloring, $\chi''_8(Shack(H_{2,2}, v, n)) \geq 10$. But with 10-coloring still not meet total r-dynamic coloring, so that plus to 11-coloring, then $\chi''_8(Shack(H_{2,2}, v, n)) \geq 11$.

To prove chromatic numbers of the total 8-dynamic coloring on graph $G = Shack(H_{2,2}, v, n)$ is 11, needs to be proven $\chi''_8(G) \geq 11$ and $\chi''_8(G) \leq 11$. Then indicated that the chromatic number $\chi''_8(G) \leq 11$ by coloring function c_{13}. Let $D = \{1, 2, 3, \ldots, k\}$ is set of colors with k colors and c_{13} is function who pairing every vertex and edges to set of color D, $c_{13} : V(Shack(H_{2,2}, v, n)) \cup E(Shack(H_{2,2}, v, n)) \to D$. Coloring function c_{13} is sa follows:

$$c_{13}(x_i) = \begin{cases} 1, \ 1 \leq i \leq n+1, \\ \quad i \equiv 1 (\text{mod } 3) \\ 6, \ 1 \leq i \leq n+1, \\ \quad i \equiv 2 (\text{mod } 3) \\ 11, \ 1 \leq i \leq n+1, \\ \quad i \equiv 0 (\text{mod } 3) \end{cases}$$

$$c_{13}(y_i) = \begin{cases} 4, \ 1 \leq i \leq n, \ i \text{ odd} \\ 5, \ 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{13}(z_i) = \begin{cases} 3, \ 1 \leq i \leq n, \ i \text{ odd} \\ 2, \ 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{13}(x_i z_i) = \begin{cases} 2, \ 1 \leq i \leq n, \ i \text{ odd} \\ 3, \ 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{13}(y_i x_{i+1}) = \begin{cases} 5, \ 1 \leq i \leq n, \ i \text{ odd} \\ 4, \ 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c_{13}(x_i x_{i+1}) = \begin{cases} 7, \ 1 \leq i \leq n, \\ \quad i \equiv 1 (\text{mod } 4) \\ 9, \ 1 \leq i \leq n, \\ \quad i \equiv 2 (\text{mod } 4) \\ 8, \ 1 \leq i \leq n, \\ \quad i \equiv 3 (\text{mod } 4) \\ 10, \ 1 \leq i \leq n, \\ \quad i \equiv 0 (\text{mod } 4) \end{cases}$$

$$c_{13}(y_i z_i) = \begin{cases} 8, \ 1 \leq i \leq n, \\ \quad i \equiv 1 (\text{mod } 4) \\ 10, \ 1 \leq i \leq n, \\ \quad i \equiv 2 (\text{mod } 4) \\ 7, \ 1 \leq i \leq n, \\ \quad i \equiv 3 (\text{mod } 4) \\ 9, \ 1 \leq i \leq n, \\ \quad i \equiv 0 (\text{mod } 4) \end{cases}$$

Of coloring function on c_{13} it is evident that the chromatic number total *shackle* graph *cocktail party* $H_{2,2}$ is $\chi''_8(G) \leq 11$. Since $\chi''_8(G) \leq 11$ and $\chi''_8(G) \geq 11$ then $\chi''_8(G) = 11$ so $\chi''_8(G) = \chi''_r(G) = 11$. As illustratio, served Figure 8 who is r-dynamic coloring of *shackle* graph *cocktail party* $H_{2,2}$.

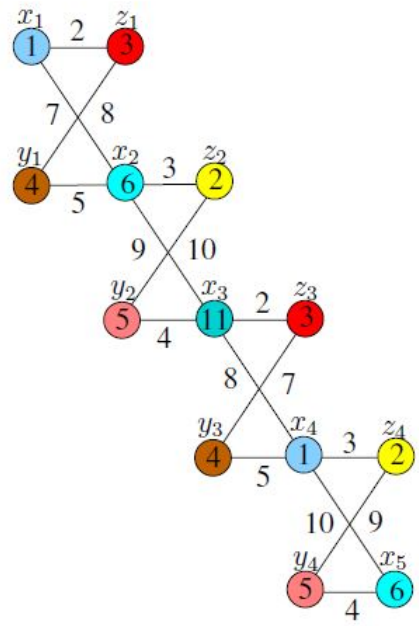

Fig 6. Total r-dynamic Coloring on Graph $Shack(H_{2,2}, v, n)$

On operation graph *shackle* graph *cocktail party* $(Shack(H_{2,2}, v, n))$, if in terms of coloring the vertex is, number of $\min\{r, \max\{d(v) + |(N(v))|\}\} = \max\{d(x_i) + |(N(x_i))|\} = 8$. In the the edge of on operation graph *shackle* graph *cocktail party* $H_{2,2}$. $(Shack(H_{2,2}, v, n))$ number of $\min\{r, \max\{d(u) + d(v)\}\} = \max\{d(u) + d(v)\} = 8$, resulting in $\chi''_{r \geq 8}(Shack(H_{2,2}, v, n)) = 11$. From the above description, so Theorem 3 proven.

□

OPEN PROBLEM

Find the chromatic number of total r-dynamic coloring of special graph and operation graph *shackle* and *generalized shackle* the other graph.

REFERENCES

[1] Slamin. "Desain Jaringan: Pendekatan Teori Graf. Jember: Universitas Jember, 2009.

[2] R. Munir, "Matematika Diskrit", 3rd Edition, Bandung: Informatika, 2009.

[3] D. B. Nugroho, "Catatan Kuliah (2 SKS) MX 324 Pengantar Teori Graf", Universitas Kristen Satya Wacana, 2008.

[4] J. Geetha and K. Somasundaran, "Total Chromatic Number and Some Topological Indices", Department of Mathematics, Amrita Vishwa Vidyapeetham, Coimbatore-641 112, India.

[5] Dafik, Hasan, Azizah and Agustin, "A Generalized Shackle of Any Graph H Admits a Super H-Antimagic Total Labelling", Article, Jember: Universitas Jember, 2010.