On The Total r-dynamic Coloring of Edge Comb Product graph $G \trianglerighteq H$

Dwi Agustin Retno Wardani[2], Dafik[1,3], Antonius C. Prihandoko[1,4], Arika I. Kristiana[1,3]

[1]CGANT University of Jember Indonesia
[2]Mathematics Depart. University of Jember Indonesia
[3]Mathematics Edu Depart. University of Jember Indonesia
[4]System Information. Depart. University of Jember Indonesia
e-mail: 2i.agustin@gmail.com

Abstract—Given that two natural numbers r, k. By a proper total k-coloring of a graph G, we mean a map $c : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$, such that any two adjacent vertices and incident edges receive different colors. A total r-dynamic coloring is a proper k-coloring c of G, such that $\forall v \in V(G), |c(N(v))| \geq min\{r, d(v) + |N(v)|\}$ and $\forall e \in E(G), |c(N(e))| \geq min\{r, d(v) + d(u)\}$. The total r-dynamic chromatic number, written as $\chi_r^{"}(G)$, is the minimum k such that G has an r-dynamic total k-coloring. A total r-dynamic coloring is a natural extension of r-dynamic coloring in which we consider more condition of the concept, namely not only assign a color on the vertices as well as on the edges. Consequently, this study will be harder. In this paper, we will initiate to analyze a total r-dynamic of an edge comb product of two graphs, denoted by $H \trianglerighteq K$, where H is path graph and K is any special graph. The result shows that the total r-dynamic chromatic number of $P_n \trianglerighteq K$.

Keywords—Total r-dynamic coloring, edge comb product of graphs, Chromatic number.

INTRODUCTION

Let G be a simple, connected and undirected graph. By a proper total k-coloring of a graph G, we mean a map $c : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$, such that any two adjacent vertices and incident edges receive different colors [1]. Azizah *et al.* in [2] defined a total r-dynamic coloring is a proper k-coloring c of G, such that

1. $\forall v \in V(G)$, $|c(N(v))| \geq min[r, d(v) + |N(v)|]$ and
2. $\forall e = uv \in E(G)$, $|c(N(e))| \geq min[r, d(v) + d(u)]$

The total r-dynamic chromatic number, written as $\chi_r^{"}(G)$, is the minimum k such that G has an r-dynamic total k-coloring. A total r-dynamic coloring is a natural extension of r-dynamic coloring in which we assign a color on the vertices as well as on the edges. In this paper, we will initiate to analyze a total r-dynamic of an edge comb product of two graphs, denoted by $H \trianglerighteq K$, where H is star graph and K is any special graph.

Saputro *et. al* firstly introduced a comb product of graph in [3]. Let H and K be two connected graphs. Let o be a vertex of K. The comb product between H and K, denoted by $H \triangleright K$, is a graph obtained by taking one copy of H and $|V(H)|$ copies of K and grafting the i-th copy of K at the vertex o to the i-th vertex of H. By the definition of comb product, we can say that $V(H \triangleright K) = \{(a, v)|a \in V(H), v \in V(K)\}$ and $(a, v)(b, w) \in E(H \triangleright K)$ whenever $a = b$ and $vw \in E(K)$, or $ab \in E(H)$ and $v = w = o$.

A natural extension of comb product of graph is an edge comb product of graph. Let H and K be two connected graphs. Let e be an edge of K. The edge comb product between H and K, denoted by $H \trianglerighteq K$, is a graph obtained by taking one copy of H and $|E(H)|$ copies of K and grafting the i-th copy of K at the edge e to the i-th edge of H. By the definition of edge comb product, we can say that $V(G \trianglerighteq H) = \{(a, v)|a \in V(G); v \in V(H)\} \cup \{(a, v, z)|a \in V(G); v, z \in V(H)\}$ and if $v = w$ and $z = y$, $y = w$ then $E(G \trianglerighteq H) = \{(a, v)(b, w, z)|a, b \in V(G); v, w, z \in V(H)\} \cup \{(b, w, z)(c, w, y)|b, c \in V(G); z, w, y \in V(H)\} \cup \{(c, w, y)(d, v)|c, d \in V(G); v, w, y \in V(H)\}$ if $a = b$ then $E(G \trianglerighteq H) = \{(a, v)(b, w)|a, b \in V(G); v, w \in V(H)\}$ so $p = |V(G \trianglerighteq H)| = q_1(p_2 - 2) + p_1$and $q = |E(G \trianglerighteq H)| = q_1 q_2$.

For $r = 1$, it is easy to see that the total 1-dynamic of any connected graph satisfies $\chi''(G) \geq \triangle(G) + 1$, where $\triangle(G)$ is the maximum degree of graph G, see [1]. Behzad and Vizing [4] also proved that the total 1-dynamic chromatic number for every graph G satisfies $\triangle(G)+1 \leq \chi''(G) \leq \triangle(G)+2$. However, we have not fixed the lower bound of the total r-dynamic of any connected graph. But the following observation holds:

Observation 1. *Let $\Delta(G)$ be a maximum degree of a graph G. the total r-dynamic of any connected graph satisfies the following $\chi''(G) \leq \chi_d''(G) \leq \chi_3''(G) \leq \cdots \leq$
$\chi_r''(G)$.*

THE RESULTS

We are ready to show our main theorems. There are two theorems found in this study that is exponential graph $P_n \trianglerighteq C_m$ and $P_n \trianglerighteq W_m$.

Theorem 1. *For $n \geq 3$, $m \geq 3$, and $r = 2\Delta$ the total r-dynamic chromatic number of edge comb product graph $G = (P_n \trianglerighteq C_m)$ is:*

$$\chi_r''(P_n \trianglerighteq C_m) = \begin{cases} \Delta + 1; & \text{for } 1 \leq r \leq \delta + 1 \\ 2\Delta; & \text{for } \delta + 2 \leq r \leq 2\Delta - 1 \\ 2\Delta + 1; & \text{for } r \geq 2\Delta \end{cases}$$

Proof. An edge comb product of path graph with cycle graph, denoted by $(\mathbf{P_n} \trianglerighteq \mathbf{C_m}), n \geq 3$ and $m \geq 3$, is a connected graph with vertex set $V(\mathbf{P_n} \trianglerighteq \mathbf{C_m}) = \{x_i; 1 \leq i \leq n\} \cup \{x_{i,j}; i \leq i \leq n-1; 1 \leq j \leq m-2\}$, and edge set $E(P_n \trianglerighteq C_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i x_{i,j}\}; 1 \leq i \leq n-1; j = 1\} \cup \{x_{i,j} x_{i,j+1}; 1 \leq i \leq n-1; 1 \leq j \leq m-3\} \cup \{x_{i,j} x_{i+1}; 1 \leq i \leq n-1; j = m-2\}$. The order and size of $(\mathbf{P_n} \trianglerighteq \mathbf{C_m}), n \geq 3, m \geq 3$ are $|V(P_n \trianglerighteq C_m)| = nm - m - n + 2, |E(P_n \trianglerighteq C_m)| = mn - m, \triangle(G) = 4$ and $\delta = 2$.

Case 1. For $\chi_r''(P_n \trianglerighteq C_m) = \Delta + 1$ it will be showed that $\chi_r''(P_n \trianglerighteq C_m) \geq \Delta + 1$, suppose $\chi_r''(P_n \trianglerighteq C_m) < \Delta + 1$ let $\chi_r''(P_n \trianglerighteq C_m) = \Delta$ then there is incident edge which has same color. As illustration see the coloring pattern with $\chi_r''(P_n \trianglerighteq C_m) = \Delta$ we can see the function:

$$c_1(x_i) = i \bmod 3;\ c_1(x_i x_{i+1}) = i + 2 \bmod 3;$$
$$c_1(x_i x_{i,1}) = 5;\ c_1(x_{i,m-2} x_i) = 4...;$$

$$c_1(x_{i,j}) = \begin{cases} 3; & \text{if } j = 1 \text{ so } i \equiv 1 \bmod 3 \\ 1; & \text{if } j \equiv 1 \bmod 2 \text{ and } j \neq 1 \text{ so } \\ & i \neq 0 \bmod 3 \text{ or} \\ & \text{if } j \equiv 0 \bmod 2 \text{ so } i \equiv 0 \bmod 3 \\ 2; & \text{if } j \equiv 0 \bmod 2 \text{ so } i \neq 0 \bmod 3 \text{ or} \\ & \text{if } j \equiv 1 \bmod 2 \text{ so } i \equiv 0 \bmod 3 \end{cases}$$

From coloring function c_1 we can see that total chromatic number of r-dynamic coloring is $\chi''_r(P_n \trianglerighteq C_m) \leq \Delta + 1$, because $\chi''_r(P_n \trianglerighteq C_m) \leq \Delta + 1$ and $\chi''_r(P_n \trianglerighteq C_m) \geq \Delta + 1$, then $\chi''_r(P_n \trianglerighteq C_m) = \Delta + 1$, so $\chi''(S_n \trianglerighteq S_m)_r = \Delta + 1; 1 \leq r \leq \delta$.

Case 2. For $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta$ it will be showed that $\chi''_r(P_n \trianglerighteq C_m) \geq 2\Delta$, suppose $\chi''_r(P_n \trianglerighteq C_m) < 2\Delta$ let $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta - 1$ then function of total coloring is c_1. Let vertex which has degree 2 is denoted by x_1 in illustration c_1 above, we can see that $|C(N(x_1)| = 3$ and $min\{r, d(x_1) + |N(x_1)|\} = min\{4, 4\} = 4$ then $3 \neq 4$ it's contradiction. So that $\chi''_r(P_n \trianglerighteq C_m) \geq 2\Delta$. Furthermore, it will be showed that $\chi''_r(P_n \trianglerighteq C_m) \leq 2\Delta$ by coloring $(P_n \trianglerighteq C_m)$ by using function c_2. Suppose $D = \{1, 2, \cdots, k\}$ is the set of k-coloring and c_2 is the which mapping the vertex and edge to D then $c_2 = V(P_n \trianglerighteq C_m) \bigcup (P_n \trianglerighteq C_m) \rightarrow \{1, 2, \cdots, k\}$ for $n \geq 3$, $m \geq 3$ and $\Delta = 4$ the function as following:

$$c_2(x_1, x_2, x_3, x_4, x_5, ...) = (1, 3, 5, 2, 4, ...);$$
$$c_2(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, ...) = (2, 4, 1, 3, 5, ...);$$
$$c_2(x_{1,1}x_{1,2}, x_{1,2}x_{1,3}, x_{1,3}x_{1,4}, ...) = (6, 7, 8, ...).$$

From coloring function c_2 we can see that total chromatic number of r-dynamic is $\chi''_r(P_n \trianglerighteq C_m) \leq 2\Delta$, because $\chi''_r(P_n \trianglerighteq C_m) \leq 2\Delta$ and $\chi''_r(P_n \trianglerighteq C_m) \geq 2\Delta$, then $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta$. So $\chi''(P_n \trianglerighteq C_m)_r = 2\Delta; \delta + 2 \leq r \leq 2\Delta - 1$.

Case 3. For $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta + 1$ it will be showed that $\chi''_r(P_n \trianglerighteq C_m) \geq 2\Delta + 1$, suppose $\chi''_r(P_n \trianglerighteq C_m) < 2\Delta + 1$ let $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta$ then function of total coloring is c_2. Let vertex which has degree 4 denoted by x_2. In illustration c_2 above, we can see that $|c(N(x_2))| = 7$ and $min\{r, d(x_2) + |N(x_2)|\} = min\{8, 8\} = 8$ then $7 \neq 8$ it's contradiction. So that $\chi''_r(P_n \trianglerighteq C_m) \geq 2\Delta + 1$. Furthermore, it will be showed that $\chi''_r(P_n \trianglerighteq C_m) \leq 2\Delta + 1$ by coloring $(P_n \trianglerighteq C_m)$ by using function c_3. Suppose $D = \{1, 2, \cdots, k\}$ is the set of k-coloring and c_3 is the function which mapping the vertex and edge to D then $c_3 = V(P_n \trianglerighteq C_m) \bigcup (P_n \trianglerighteq C_m) \rightarrow \{1, 2, \cdots, k\}$ for $n \geq 3$, $m \geq 3$ and $\Delta = 4$ the function as following:

$$c_3(x_1, x_2, x_3, x_4, x_5, ...) = (1, 3, 5, 2, 4, ...);$$
$$c_3(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, ...) = (2, 4, 1, 3, 5, ...);$$
$$c_3(x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, ...) = (6, 7, 8, 9, ...).$$

From coloring function c_3 we can see that total chromatic number of r-dynamic is $\chi''_r(P_n \trianglerighteq C_m) \leq 2\Delta + 1$, because $\chi''_r(P_n \trianglerighteq C_m) \leq 2\Delta + 1$ and $\chi''_r(P_n \trianglerighteq C_m) \geq 2\Delta + 1$, then $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta + 1$, so $\chi''_r(P_n \trianglerighteq C_m) = 2\Delta + 1$. In the exponential graph $(P_n \trianglerighteq C_m)$ the value of $min\{r, max\{d(u) + d(v)\}\} = max\{d(u) + d(v)\} = 2\Delta + 1$ and $min\{r, max\{d(u) + |N(u)|\}\} = max\{d(u) + |N(u)|\} = 2\Delta + 1$ resulting $\chi''_{r \geq 2\Delta}(P_n \trianglerighteq C_m) = 2\Delta + 1$.

So the Theorem is proved. □

Theorem 2. *For $n \geq 3$, $m \geq 3$, and $r = 2\Delta$ the total r-dynamic chromatic number of edge comb product graph $G = (P_n \trianglerighteq W_m)$ is:*

$$\chi''_r(P_n \trianglerighteq W_m) = \begin{cases} \Delta + 1; & \text{for } 1 \leq r \leq \delta + 1 \\ r + 1; & \text{for } \delta + 2 \leq r \leq 2\Delta - 2 \\ 2\Delta + 1; & \text{for } r \geq 2\Delta - 1 \end{cases}$$

Proof. An edge comb product of path graph with wheel graph, denoted by $(\mathbf{P_n} \trianglerighteq \mathbf{W_m}), n \geq 3$ and $m \geq 3$, is a connected graph with vertex set $V(\mathbf{P_n} \trianglerighteq \mathbf{W_m}) = \{x_i; 1 \leq i \leq n\} \cup \{A_i; 1 \leq i \leq n-1\} \cup \{x_{i,j}; i \leq i \leq n-1; 1 \leq j \leq m-2\}$, and edge set $E(P_n \trianglerighteq W_m) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{A_i x_i; 1 \leq i \leq n-1\} \cup \{A_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{A_i x_{i,j}; 1 \leq i \leq n-1; 1 \leq j \leq m-2\} \cup \{x_i x_{i,j}\}; 1 \leq i \leq n-1; j = 1\} \cup \{x_{i,j} x_{i,j+1}; 1 \leq i \leq n-1; 1 \leq j \leq m-3\} \cup \{x_{i,j} x_{i+1}; 1 \leq i \leq n-1; j = m-2\}$. The order and size of $(\mathbf{P_n} \trianglerighteq \mathbf{W_m}), n \geq 3, m \geq 3$ are $|V(P_n \trianglerighteq W_m)| = nm - m + 1$, $|E(P_n \trianglerighteq W_m)| = 2(mn - m)$, $\delta = 3$ and $\begin{cases} 6; & \text{for } m \leq 6 \\ m; & \text{for } m \geq 6 \end{cases}$

CONCLUDING REMARKS

We have found that some total $r-$dynamic chromatic number of exponential graph $P_n \trianglerighteq H$. It is interesting to characterize a property of any graph operation to have an exact value or lower bound of their total $r-$dynamic chromatic numbers.

Conjecture 1. *Let Δ be maximum degree of graph $P_n \trianglerighteq H$. The upper bound of total r-dynamic chromatic number of $P_n \trianglerighteq H$ is $\chi''(P_n \trianglerighteq H) \leq 2\Delta + 1$. It is sharp.*

Note. Let $v \epsilon V(G)$ with v as maximum degree. According to Kowalik [5] and [6], all of graphs are limited, simple and undirected. If it let $G = (V(G), E(G))$ is a graph with vertex set $(V(G)$ and edges set $E(G)$. Total coloring of graph G is mapping $c : V(G) \cup E(G) \rightarrow [1, 2, 3..., k]$ where $[1, 2, 3, ..., k]$ is a color set that complete this condition:

1. $c(u) \neq c(v)$, for every two vertices that adjacent, where $u, v \epsilon V(G)$;
2. $c(uv) \neq c(u'v)$, for every two two edges that adjacent, where $uv, u'v \epsilon E(G)$;
3. $c(v) \neq c(uv)$, for every vertex $v \epsilon V(G)$ and other edges $uv \epsilon E(G)$ incident in vertex v.

From the statement 1 we found $\chi''_r \geq \Delta + 1$ so that every adjacent vertices must have different colors. From statement 2, we see that every adjacent edges must have different colors to v where v is maximum degree. So that:

$\chi''_r \geq \Delta + 1 = \Delta + 1 + \Delta$
$\chi''_r \geq \Delta + 1 = 2\Delta + 1$ □

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