The Rainbow (1,2)-Connection Number of Edge Comb Product Graph and It's Lower Bound

Gembong A.W. [3],Dafik[1,2], Ika Hesti Agustin[1,3], Slamin [1,4]
[1]CGANT University of Jember Indonesia [2]Mathematics
Edu. Depart. University of Jember Indonesia
[3]Mathematics Depart. University of Jember Indonesia
[4]Information System Depart. University of Jember Indonesia
e-mail: gembongangger@gmail.com

Abstract—Let $G = (V, E)$ be a simple, nontrivial, finite, connected and undirected graph. Let c be a coloring $c : E(G) \to \{1, 2, \dots, k\}, k \in$ N. A path in an edge colored graph is said to be a rainbow path if no two edges on the path have the same color. An edge colored graph G is rainbow connected if there exists a rainbow $u - v$ path for every two vertices u and v of G. The rainbow connection number of a graph G, denoted by $rc(G)$, is the smallest number of k colors required to edge color the graph such that the graph is rainbow connected. Furthermore, for an l-connected graph G and an integer k with $1 \leq k \leq l$, the rainbow k-connection number $rc_k(G)$ of G is defined to be the minimum number of colors required to color the edges of G such that every two distinct vertices of G are connected by at least k internally disjoint rainbow paths. In this paper, we determine the exact values of rainbow connection number of exponential graphs, namely Path of ladder as exponent, Cycle of Ladder as exponent, Cycle of Triangular Ladder as exponent, Cycle of Complete as exponent. We also proved that $rc_2(G) = diam(G) + 1$.

Keywords—Rainbow l-Connection Number, Graph Operations.

INTRODUCTION

Suppose G is a simple connected graph with a set of points $V(G)$ and edge $E(G)$. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \dots, k\}, k \in N$, of the edges of G, where adjacent edges may be colored the same. A $u - v$ path P in G is a rainbow path if no two edges of P are colored the same. The graph G is rainbow-connected (with respect to c) if G contains a rainbow $u - v$ path for every two vertices u and v of G. In this case, the coloring c is called a rainbow coloring of G. If k colors are used, then c is a rainbow k-coloring. The minimum k for which there exists a *rainbow k-coloring* of the edges of G is the rainbow connection number $rc(G)$ of G. A rainbow coloring of G using $rc(G)$ colors is called a minimum rainbow coloring of G. This definition can be find in Chartrand in [1].

An easy observation is that if G has n vertices then $rc(G) \leq n - 1$, since one may color the edges of a given spanning tree with distinct colors, and color the remaining edges with one of the already used colors or leave the remaining edges uncolored, Caro in [2]. Also notice that, clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G, Caro in [2]. So we have:

$$diam(G) \leq rc(G) \leq n - 1 \tag{1}$$

A well-known result shows that in every $l-$connected graph G with $l \geq 1$, there are k internally disjoint $u - v$ paths connecting any two distinct vertices u and v for every integer k with $1 \leq k \leq l$ [3]. Chartrand et al. [4] defined the rainbow k-connectivity $rc_k(G)$ of G to be the minimum integer j for which there exists a j-edge-coloring of G such that for every two distinct vertices u and v of G, there exist at least k internally disjoint $u-v$ rainbow paths.

By the definition of rainbow k-connectivity $rc_k(G)$, we know that it is almost impossible to derive the exact value or a nice bound of the rainbow k-connectivity for a general graph G [3]. So one investigates the rainbow k-connectivity of some classes of special graphs. In this article we discuss $rc(G)$ for G is the graph operation Path Powers ladder, Cycle Powers Triangular Ladder, and Cycle Powers Complete with the order of Cycle is even, all its $rc(G)$ value is filled by diameter. For $rc_2(G)$ we find that the graph with $rc_2(G) = diam(G) + 1$ filled by Cycle Powers Triangular Ladder and Cycle Powers Complete when the order of Cycle is four. We determine $rc_k(G)$ by using edge coloring function. Edge coloring function is a set $\{1, 2, \dots, n\}$ so we write $0 \equiv \text{mod } b$ as $b \equiv \text{mod } b$.

THE RESULTS

In this section we proof that $rc(G) = diam(G)$ for G is fan, path edge comb ladder, and cycle edge comb cycle. For $rc_2(G)$ we proof that $rc_2(G) = diam(G) + 1$.

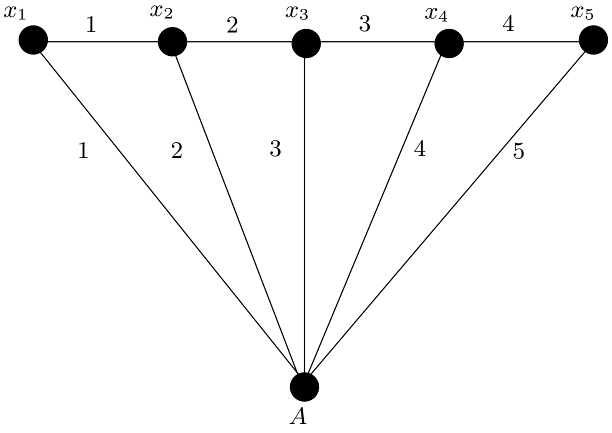

Fig 1. Graph $G = F_5$ with $rc_2(G) = 5$

Theorem 1. *Let G be a fan graph, its rainbow 2-connection number is $rc_2(G) = n$.*

Proof. Suppose $G = F_n$. The graph G has vertex set $V(G) = \{x_i; 1 \leq i \leq n\} \cup \{A\}$ and edge set $E(G) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{Ax_i; 1 \leq i \leq n\}$. Define a color c of the edges $c : E(G) \to \{1, 2, \dots, k\}, k \in N$:

$$c(e) = i, e \in \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{Ax_i; 1 \leq i \leq n\}$$

It is easy to see that the color $c(e)$ reach a maximum value when $e = Ax_n$ and $c(e) = n$. Thus, $rc_2(G) \leq n$. Based on Theorem 1, the lower bound of rainbow 2-connection number of G is $rc_2(G) \geq (n + 1) - 2 = n - 1$. However, we can not attain this sharpest lower bound. Consider edge set $E' = \{x_i x_{i+1} | 1 \leq i \leq n-1\}$ and $E" = \{Ax_i | 1 \leq i \leq n\}$. If we color n edges of $E"$ by $n - 1$ colors, then there exist $e_1, e_2 \in E'$ such that $c(e_1) = c(e_2)$, without loss of generality we can choose $e_1 = Ax_1$ and $e_2 = Ax_n$. Since F_n is 2-connected graph and $rc_2(F_n) = n - 1$ then there must exist two disjoint paths between any two vertices. Consider vertex x_1 and vertex x_n which give two disjoint paths between x_1 and x_n. The first possible rainbow path is $x_1 x_2 \dots x_{n-1} x_n$, the second is $x_1 A x_n$, for x_1, A and x_n is not rainbow path as $c(x_1 A) = c(Ax_n)$. Thus, we have lower bound of rainbow 2-connection number of G is $rc_2(G) \geq n$. It concludes that $rc_2(G) = n$.

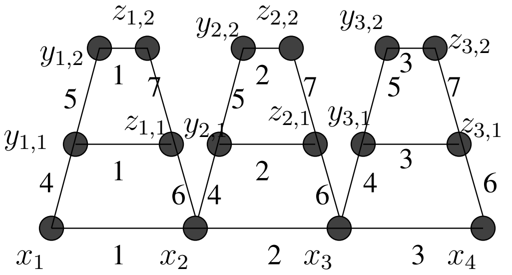

Fig 2. Edge comb product graph $P_4^{L_3}$ with $rc(G) = 7$

Theorem 2. *If $G = P_n \trianglerighteq L_m$, then the numbers $rc(G) = 2m + n - 3$.*

Proof. Graph $G = P_n \trianglerighteq L_m$ is graph with cardinality: $V(G) = \{x_i | 1 \leq i \leq n\} \cup \{y_{i,j} | 1 \leq i \leq n-1, 1 \leq j \leq m-1\} \cup \{z_{i,j} | 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$ and $E(G) = \{x_i x_{i+1} | i \leq i \leq n-1\} \cup \{x_i y_{i,1} | 1 \leq i \leq n-1\} \cup \{x_{i+1} z_{i,1} | 1 \leq i \leq n-1\} \cup \{y_{i,j} y_{i,j+1} | 1 \leq i \leq n-1, 1 \leq j \leq m-2\} \cup \{z_{i,j} z_{i,j+1} | 1 \leq i \leq n-1, 1 \leq j \leq m-2\} \cup \{y_{i,j} z_{i,j} | 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$. The value of $|V(G)| = n + 2(n-1)(m-1)$ and $|E(G)| = (n-1)(3m-2)$. The diameter of G, $diam(G) = 2m + n - 3$. Number $rc(G)$ is given by the following function:

$$c(e) = \begin{cases} i, & e \in \{x_i x_{i+1} | i \leq i \leq n \\ & -1\} \cup \{y_{i,j} z_{i,j} | 1 \leq i \leq \\ & n-1, 1 \leq j \leq m-1\} \\ n, & e \in \{x_i y_{i,1} | 1 \leq i \leq n \\ & -1\} \\ n+j, & e \in \{y_{i,j} y_{i,j+1} | 1 \leq i \leq \\ & n-1, 1 \leq j \leq m-2\} \\ n+m-1, & e \in \{x_{i+1} z_{i,1} | 1 \leq i \leq \\ & n-1\} \\ n+m-1+j, & e \in \{z_{i,j} z_{i,j+1} | 1 \leq i \leq \\ & n-1, 1 \leq j \leq m-2\} \end{cases}$$

The maximun value of $c(e) = 2m + n - 3$ or $rc(G) \leq 2m + n - 3$ and by Inequality 1 $rc(G) \geq 2m + n - 3$ so $rc(G) = 2m + n - 3$. We cannot determine $rc_2(G)$ because $G = P_n \trianglerighteq L_m$ is 1-connected graph.

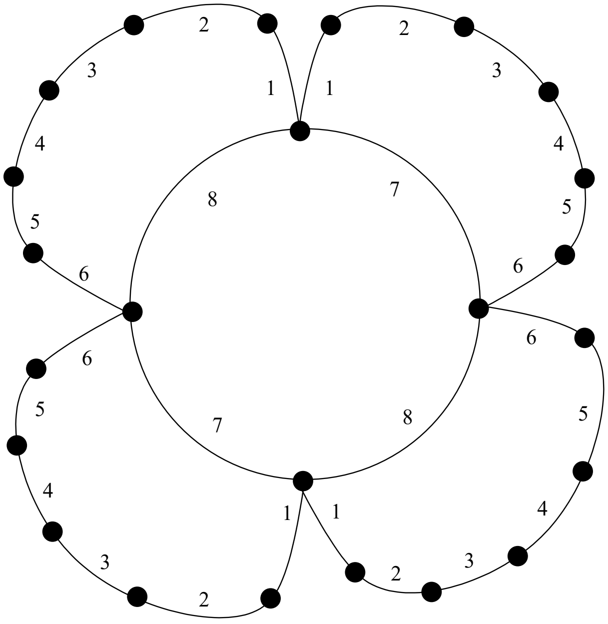

Fig 3. Edge comb product graph $C_4^{C_m}$ with $rc(G) = 6$

Theorem 3. *If $G = C_4 \trianglerighteq C_m$, then the numbers $rc(G) = m + 1$ for m odd and $rc(G) = m + 2$ for m even.*

Proof. Graph $G = C_4 \trianglerighteq C_m$ is graph with cardinality: $V(G) = \{x_i | 1 \leq i \leq 4\} \cup \{x_{ij} | 1 \leq i \leq 4, 1 \leq j \leq m-2\}$ and $E(G) = \{x_i x_{i+1} | i \leq i \leq 3\} \cup \{x_1 x_4\} \cup \{x_j^i x_{j+1}^i | 1 \leq i \leq 4, 1 \leq j \leq m-3\} \cup \{x_i x_1^i | 1 \leq i \leq 4\} \cup \{x_{i+1} x_{m-2}^i | 1 \leq i \leq 3\} \cup \{x_1 x_{m-2}^4\}$. The value of $|V(G)| = 4m - 4$ and $|E(G)| = 4m$. The diameter of G, $diam(G) = 2m + n - 3$. Number $rc(G)$ is given by the following function for m odd:

$$c(e) = \begin{cases} m+1, & e \in \{x_2 x_3 \cup x_4 x_1\} \\ m, & e \in \{x_1 x_2 \cup x_3 x_4\} \\ 1, & e \in \{x_i x_1^i | 1 \leq i \leq 4\} \\ m-1, & e \in \{x_{i+1} x_{m-2}^i | 1 \leq i \leq 4 \cup x_1 \\ & x_{m-2}\} \\ j+1, & e \in \{x_j^i x_{j+1}^i | 1 \leq i \leq 4, 1 \leq j \leq \\ & m-3\} \end{cases}$$

The maximun value of $c(e) = m + 1$ or $rc(G) \leq m + 1$ and by Inequality 1 $rc(G) \geq m + 1$ so $rc(G) = m + 1$.

$$c(e) = \begin{cases} m, & e \in \{x_1 x_2 \cup x_3 x_4 \cup x^1_{\frac{m-2}{2}} x_{\frac{m}{2}1}\} \\ m+2, & e \in \{x^2_{\frac{m-2}{2}} x^2_{\frac{m}{2}}\} \\ \frac{m}{2}+1, & e \in \{x^3_{\frac{m-2}{2}} x^3_{\frac{m}{2}}\} \\ m+1, & e \in \{x_2 x_3 \cup x_1 x_4 \cup x^4_{\frac{m-2}{2}} x^4_{\frac{m}{2}}\} \\ 1, & e \in \{x_i x_1^i | 1 \leq i \leq 4\} \\ m-1, & e \in \{x_{i+1} x_{m-2}^i | 1 \leq i \leq 3 \cup \\ & x_1 x_{m-2}^4\} \\ j+1, & e \in \{x_j^i x_{j+1}^i | 1 \leq i \leq 4, 1 \leq j \leq \\ & \frac{m-2}{2} - 1, \frac{m}{2} \leq j \leq m-2\} \end{cases}$$

The maximun value of $c(e) = m + 2$ or $rc(G) \leq m + 2$ and by Inequality 1 $rc(G) \geq m + 2$ so $rc(G) = m + 2$.

CONCLUSIONS

In this paper, we have presented the number of $rc(G)$ and $rc_2(G)$ for graph operation of some special graph. The number $rc(G)$ for G is the graph operation Path Powers ladder, Cycle Powers Triangular Ladder, and Cycle Powers Complete with the order of Cycle is even, all its $rc(G)$ value is filled by diameter. For $rc_2(G)$ we find that the graph with $rc_2(G) = diam(G) + 1$ filled by Cycle Powers Triangular Ladder and Cycle Powers Complete when the order of Cycle is four. Also in this paper, we proposed a lower bound for $rc_2(G)$: max $\{|C(u,v)| - d(u,v)\}$, for $C(u,v)$ is cycle that contain any vertex u and v in $V(G)$.

Besides the result we also, have a question as open problem: "Do $rc(C_n \trianglelefteq H) > diam(C_n \trianglelefteq H)$ for H be any simple connected graph and n is odd and greater than 4?" or in general we want to know: "Is there any relation between the size of cycle in G with the value $rc(G)$ (greater or equal to diameter)?".

REFERENCES

[1] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, *Math. Bohem.* **133**, 85-98, 2008

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, *Electron. J. Combin.* **15**, R57, 2008

[3] Xueliang Li, Yuefang Sun, On the rainbow k-connectivity of complete graphs, Australian Journal Of Combinatorics, 217-226, 2011

[4] G. Chartrand, G.L. Johns, K.A.McKeon and P. Zhang, The rainbow connectivity of a graph, Networks 54, 75-81, 2009