THE METRIC COLORING OF RELATED WHEEL GRAPHS

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ABSTRACT

In this paper, graphs are finite and connected graph. Let $f: V(G) \rightarrow \{1, 2, \ldots, k\}$ be a vertex coloring of a graph G where two adjacent vertices may be colored the same color. Consider the color classes $\pi = \{C_1, C_2, \ldots, C_k\}$. For a vertex $v \in G$, representation color of v is the k-vector $r(v|\pi) = (d(v,, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = min\{d(v, c);\ c \in C_i\}$. If $r(u \mid \Pi) \neq r(v \mid \Pi)$ for every two adjacent vertices u and v of G, then f is called a metric coloring of G. The minimum k for which G has a metric k-coloring is called the metric chromaticnumber of G and is denoted by $\mu(G)$. In this paper, we study the metric chromatic numbers of related wheel graphs namely double wheel graph, web graph, friendship graph and helm graph.

Key words : metric coloring, related wheel graph.

INTRODUCTION

We consider finite and connected graphs. Let $G = (V, E)$ be a graph G with the vertex set, $V(G)$ and the edge set $E(G)$. The distance between two distinct vertices $u, v \in V(G)$ is the length of a shortest uv path in G, denotedby $d(u, v)$. The concept introduced by Chartrand, et.al [2], [5]. Suppose that $f: V(G) \rightarrow \{1, 2, \ldots, k\}$ is a k-coloring of G where two adjacent vertices may be colored the same color. Consider the color classes $\pi\{C_1, C_2, \ldots, C_k\}$. For a vertex $v \in G$, representation color of v is the k-vector $r(v|\Pi) = (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$ where $d(v, C_i) = min\{d(v, c); c \in C_i\}$. If $r(u|\Pi) \neq r(v|\Pi)$ for every two adjacent vertices u and v of G, then f is called a metric coloring of G. The minimum k for which G has a metric k-coloring is called the metric chromatic number of G and is denoted by $\mu(G)$. This definition introduced by Zhang, et.al [3].

Alfarisi, et.al [1] founded the metric chromatic number of unicyclic graph. Futhermore, Rohmatulloh, et.al [4] obtained the metric chromatic number of comb product of ladder graph. Two proposition of the metric chromatic number introduced by Zhang, et.al [3] as follows.

Proposition 1.1. *Let G be a connected graph of order n,* $2 \leq \mu(G) \leq \chi(G) \leq n$.

Proposition 1.2. *If G is a connected graph with $\chi(G) = 3$, then* $\mu(G) = 3$.

RESULTS

We study the metric coloring of related wheel graphs, namely double wheel graph, web graph, friendship graph and helm graph.

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Theorem 2.1. *Consider wheel graph W_n for $n \geq 3$, then* $\mu(W_n) = 3$.
Proof. $V(W_n) = \{a, ai; 1 \leq i \leq n\}$ and $E(W_n) = \{a, a_i; 1 \leq i \leq n\} \cup \{a, a_{(i+1)}; 1 \leq i \leq n-1\} \cup \{a_1 a_n\}$. Based on Proposition 1, $\mu(W_n) \geq 2$. Thus, this proof divided into two cases as follows.

Case 1. For n is even

We prove that $\mu(W_n) \leq 3$. Let $f: V(W_n) \to \{1, 2, 3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph W_n with the periodic label in cycle $(2, 3, 2, 3, 2, 3, 2, 3, \ldots, 2, 3)$ and 1 for center of wheel graph. Furthermore, the label color and the representation of vertices in wheel graph W_n respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a\}$, $C_2 = \{a_i;\ i\ is\ odd\}$ and $C_3 = \{a_i;\ i\ is\ even\}$ as follows.

$$f(x) = \begin{cases} 1, & v = a \\ 2, & v \in a_i, i \text{ odd} \\ 3, & v \in a_i, i \text{ even} \end{cases}$$

Based on the color label f in wheel graph W_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0, 1, 1);$$
$$r(a_i|\Pi) = (1, 0, 1), for\ i\ is\ odd;$$
$$r(a_i|\Pi) = (1, 1, 0), for\ i\ is\ even.$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a_i|\Pi) \neq r(a_{(i+1)}|\Pi)$ or $r(a|\Pi) \neq r(a_i|\Pi)$. We obtain $\mu(W_n) \leq 3$. Hence, $\mu(W_n) = 3$

Case 2. For n is odd

We prove that $\mu(W_n) \leq 3$. Let $f: V(W_n) \to \{1, 2, 3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph W_n with the periodic label in cycle $(1, 2, 3, 2, 3, 2, 3, \ldots, 2, 3, 1, 1)$ and 1 for center of wheel graph. Furthermore, the label color and the representation of vertices in wheel graph W_n respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a, a_{(n-1)}, a_n\}$, $C_2 = \{a_i;\ i\ is\ even, 2 \leq i \leq n-2\}$ and $C_3 = \{a_i;\ i\ is\ odd, 2 \leq i \leq n-2\}$ as follows.

$$f(x) = \begin{cases} 1, & v \in \{a, a_{(n-1)}, a_n\} \\ 2, & v \in a_i, i \text{ even}, 2 \leq i \leq n-2 \\ 3, & v \in a_i, i \text{ odd}, 2 \leq i \leq n-2 \end{cases}$$

Based on the color label f in wheel graph W_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0, 1, 1);$$
$$r(a|\Pi) = (0, 1, 1);$$
$$r(a_1|\Pi) = (0, 1, 2);$$

$$r\left(a_{(n-1)}\middle|\Pi\right) = (0, 2, 1);$$
$$r(a_n|\Pi) = (0, 2, 2);$$
$$r(a_i|\Pi) = (1, 0, 1), for\ i\ is\ even, 2 \le i \le n - 2;$$
$$r(a_i|\Pi) = (1, 1, 0), for\ i\ is\ odd, 2 \le i \le n - 2.$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a_i|\Pi) \neq r(a_{(i+1)}|\Pi)$ or $r(a \mid \Pi) \neq r(a_i \mid \Pi)$. We obtain $\mu(W_n) \le 3$. Hence, $\mu(W_n) = 3$.

Lemma 2.1. *Let G be related wheel graph and W_n be wheel graph, then* $\mu(G) \ge \mu(W_n)$
Proof. To prove this lemma, we will illustrate by describing the following conditions:

- (i). Let G be related wheel graph such that K subgraph G, where K be rim of wheel graph. Rim wheel graph is cycle order three, C_3 then it has 3 different colors;
- (ii). Based on Theorem 1, $\mu(W_n) = 3$;
- (iii). Because G related wheel graph then there is a possibility that the order and size of G is bigger than the wheel graph.

By (i), (ii) and (iii), we get that $\mu(G) \ge \mu(W_n)$. It completes the proof.

Some graphs of related wheel graph are web graph, double wheel graph, friendship graph and helm graph.

Theorem 2.2. *Consider web graph Wb_n for $n \ge 3$, then* $\mu(Wb_n) = 3$.
Proof. $V((Wb)_n) = \{a, a_i, b_i, c_i; 1 \le i \le n\}$ and $E((Wb)_n) = \{aa_i; 1 \le i \le n\} \cup \{a_ib_i; 1 \le i \le n\} \cup \{a_ia_{(i+1)}; 1 \le i \le n-1\} \cup \{a_1a_n\} \cup \{b_ib_{(i+1)}; 1 \le i \le n-1\} \cup \{b_1b_n\} \cup \{b_ic_i; 1 \le i \le n\}$. Based on Lemma 2.1, $\mu((Wb)_n) \ge \mu(W_n) = 3$. Thus, we prove that $\mu((Wb_n) \le 3$. Let $f: V\ ((Wb)_n) \to \{1, 2, 3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph Wb_n with the periodic label in cycle (2,3,2,3,2,3,…,2,3), (1,1,1,1,1,1,…,1,1) in outer cycle and 1 for center of web graph. Furthermore, the label color and the representation of vertices in web graph (Wb_n) respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a, b_i, c_i\}$, $C_2 = \{a_i; i\ is\ odd\}$ and $C_3 = \{a_i; i\ is\ even\}$ as follows.

$$f(x) = \begin{cases} 1, & v \in \{a, b_i,\ c_i\} \\ 2, & v \in a_i, i \text{ odd} \\ 3, & v \in a_i, i \text{ even} \end{cases}$$

Based on the color label f in web graph Wb_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0, 1, 1);$$
$$r(b_i|\Pi) = (0, 1, 2), for\ i\ is\ odd;$$
$$r(b_i|\Pi) = (0, 2, 1), for\ i\ is\ even;$$
$$r(a_i|\Pi) = (1, 0, 1), for\ i\ is\ odd;$$
$$r(a_i|\Pi) = (1, 1, 0), for\ i\ is\ even;$$
$$r(c_i|\Pi) = (0, 2, 3), for\ i\ is\ odd;$$
$$r(c_i|\Pi) = (0, 3, 2), for\ i\ is\ even.$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a_i|\Pi) \neq r(a_{(i+1)}|\Pi)$ or $r(b_i|\Pi) \neq r(b_{(i+1)}|\Pi)$. We obtain $\mu(Wb_n) \leq 3$. Hence, $\mu(Wb_n) = 3$. It completes the proof.

Theorem 2.3. *Consider double wheel graph DW_n for $n \geq 3$, then* $\mu(DW_n) = 3$.
Proof. $V((DW)_n) = \{a, a_i, b_i; 1 \leq i \leq n\}$ and $E((DW)_n) = \{aa_i; 1 \leq i \leq n\} \cup \{a_ib_i; 1 \leq i \leq n\} \cup \{a_ia_{(i+1)}; 1 \leq i \leq n-1\} \cup \{a_1a_n\} \cup \{b_ib_{(i+1)}; 1 \leq i \leq n-1\} \cup \{b_1b_n\}$. Based on Lemma 2.1, $\mu((DW)_n) \geq 3$. Thus, we prove that $\mu((DW)_n) \leq 3$. Let $f: V((DW)_n) \to \{1, 2, 3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph DW_n with the periodic label in cycle $(2,1,2,1,2,1,\ldots,2,1)$, $(1,1,1,1,1,1,\ldots,1,1)$ in outer cycle and 1 for center of double wheel graph. Furthermore, the label color and the representation of vertices in double wheel graph (DW_n) respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a, b_i\}$, $C_2 = \{a_i,; \ i \ is \ odd\}$ and $C_3 = \{a_i; \ i \ is \ even\}$ as follows.

$$f(x) = \begin{cases} 1, & v \in \{a, bi\} \\ 2, & v \in a_i, i \text{ odd} \\ 3, & v \in a_i, i \text{ even} \end{cases}$$

Based on the color label f in double wheel graph DW_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0, 1, 1);$$
$$r(b_i|\Pi) = (0, 1, 2), for\ i\ is\ odd;$$
$$r(b_i|\Pi) = (0, 2, 1), for\ i\ is\ even;$$
$$r(a_i|\Pi) = (1, 0, 1), for\ i\ is\ odd;$$
$$r(a_i|\Pi) = (1, 1, 0), for\ i\ is\ even.$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a_i|\Pi) \neq r(a_{(i+1)}|\Pi)$ or $r(b_i|\Pi) \neq r(b_{(i+1)}|\Pi)$. We obtain $(DW_n) \leq 3$. Hence, $\mu(DW_n) = 3$. It completes the proof.

Theorem 2.4. *Consider friendship graph F_n for $n \geq 3$, then* $\mu(F_n) = 3$.
Proof. $V(F_n) = \{a, a_i; 1 \leq i \leq 2n\}$ and $E(F_n) = \{aa_i; 1 \leq i \leq 2n\} \cup \{a_ia_{(i+1)}; \ i \ is \ odd, 1 \leq i \leq 2n-1\}$. Based on Lemma 2.1, $\mu(F_n) \geq \mu(W_n) = 3$. Thus, we prove that $\mu(F_n) \leq 3$. Let $f: V\ (F_n) \to \{1, 2, 3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph F_n with the periodic label in cycle $(1,2,1,2,1,2,1,2,\ldots,1,2)$ in outer cycle and 3 for center of friendship graph. Furthermore, the label color and the representation of vertices in web graph F_n respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a\}, C_2 = \{a_i,; \ i \ is \ odd\}$ and $C_3 = \{a_i; \ i \ is \ even\}$ as follows.

$$f(x) = \begin{cases} 1, & v \in a_i, i \text{ odd}, 1 \leq i \leq 2n \\ 2, & v \in a_i, i \text{ even}, 1 \leq i \leq 2n \\ 3, & v = a \end{cases}$$

Based on the color label f in friendship graph F_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0,1,1);$$
$$r(a_i|\Pi) = (1,0,1), for\ i\ is\ even;$$
$$r(a_i|\Pi) = (0,1,1), for\ i\ is\ odd.$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a|\Pi) \neq r(a_i|\Pi)$. We obtain $\mu(F_n) \leq 3$. Hence, $\mu(F_n) = 3$. It completes the proof.

Theorem 2.5. *Consider helm graph* H_n *for* $n \geq 3$, *then* $\mu(H_n) = 3$.
Proof. $V(H_n) = \{a, a_i, b_i; 1 \leq i \leq n\}$ and $E(H_n) = \{aa_i; 1 \leq i \leq n\} \cup \{a_i a_{(i+1)}; 1 \leq i \leq n-1\} \cup \{a_1 a_n\} \cup \{a_i b_i; 1 \leq i \leq n\}$. Based on Lemma 2.1, $\mu(H_n) \geq \mu(W_n) = 3$. Thus, this proof divided into two cases as follows.

Case 1. For *n* is even

We prove that $\mu(H_n) \leq 3$. Let $f: V(H_n) \rightarrow \{1,2,3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph H_n with the periodic label in cycle (1,2,1,2,1,2,1,2, . . . ,1,2) in outer cycle and 3 for center of helm graph. Furthermore, the label color and the representation of vertices in helm graph H_n respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a, b_i\}$, $C_2 = \{a_i,; i\ is\ odd\}$ and $C_3 = \{a_i;\ i\ is\ even\}$ as follows.

$$f(x) = \begin{cases} 1, & v \in \{a,\ b_i\} \\ 2, & v \in a_i, i \text{ odd} \\ 3, & v \in a_i, i \text{ even} \end{cases}$$

Based on the color label f in helm graph H_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0,1,1);$$
$$r(a_i|\Pi) = (1,0,1), for\ i\ is\ odd;$$
$$r(a_i|\Pi) = (1,1,0), for\ i\ is\ even;$$
$$r(b_i|\Pi) = (0,1,2), for\ i\ is\ odd;$$
$$r(b_i|\Pi) = (0,2,1), for\ i\ is\ even.$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a_i|\Pi) \neq r(a_{\ (i+1)}|\Pi)$ or $r(a_i|\Pi) \neq r(b_i|\Pi)$. We obtain $\mu(H_n) \leq 3$. Hence $\mu(H_n) = 3$ for n is even.

Case 2. For *n* is odd

We prove that $\mu(H_n) \leq 3$. Let $f: V\ (F_n) \rightarrow \{1,2,3\}$ be a vertex coloring (two adjacent vertices may be colored same color). The construction of coloring in graph H_n with the periodic label in cycle (1,2,3,2,3,2,3, . . ,2,3,1,1), in pendant (1,1,1,1, . . . ,1,1) and 1 in outer cycle of helm graph. Furthermore, the label color and the representation of vertices in helm graph H_n respect to with class color $\Pi = \{C_1, C_2, C_3\}$ where $C_1 = \{a, a_{(n-1)}, a_n, b_i\}$, $C_2 = \{a_i,; i\ is\ even, 2 \leq i \leq n-2\}$ and $C_3 = \{a_i; i\ is\ even, 2 \leq i \leq n-2\}$ as follows.

$$f(x) = \begin{cases} 1, & v \in \{a,\ a_{(n-1)}, a_n,\ b_i\} \\ 2, & v \in a_i, i \text{ even}, 2 \le i \le n-2 \\ 3, & v \in a_i, i \text{ odd}, 2 \le i \le n-2 \end{cases}$$

Based on the color label f in helm graph H_n. Thus, we have the representation as follows.

$$r(a|\Pi) = (0,1,1);$$
$$r(a_1|\Pi) = r(b_i|\Pi) = (0,1,2);\ for\ i\ is\ even, 2 \le i \le n-2;$$
$$r\big(a_{(n-2)}\big|\Pi\big) = r(b_i|\Pi) = (0,2,1);\ for\ i\ is\ odd, 2 \le i \le n-2;$$
$$r(a_n|\Pi) = (0,2,2)$$
$$r(a_i|\Pi) = (1,0,1);\ for\ i\ is\ even, 2 \le i \le n-2;$$
$$r(a_i|\Pi) = (1,1,0);\ for\ i\ is\ odd, 2 \le i \le n-2;$$
$$r(b_1|\Pi) = r\big(b_{(n-1)}\big|\Pi\big) = (0,2,3);$$
$$r(b_n|\Pi) = (0,3,3);$$

Clearly, for every two adjacent vertices has distinct representation, we can see in $r(a_i|\Pi) \neq r(a_{(i+1)}|\Pi)$ or $r(a_i|\Pi) \neq r(b_i|\Pi)$. We obtain $\mu(H_n) \le 3$. Hence $\mu(H_n) = 3$ for n is odd. It completes the proof.

CONCLUSION

In this paper we have shown some the exact values metric chromatic number of related wheel graphs, namely web graph, double wheel graph, friendship graph and helm graph.

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