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# *ON CHROMATIC POLYNOMIAL OF A FAN GRAPH*

**(Polinomial Kromatik pada Graf Kipas)**

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**Abstract.** A chromatic polynomial of a graph $G$ is a special function that describes the number of ways we can achieve a proper coloring on the vertices of $G$ given $k$ colors. In this paper, we determine a chromatic polynomial of a fan graph.

**Keywords:** Proper coloring, chromatic polynomial, fan graph.
**MSC 2010:** 05C31, 05C15

## 1. Introduction

Let $G$ be a simple labelled graph. A proper coloring of a graph $G$ is an assignment of colors to each vertex of $G$ such that no edge connects two identically colored vertices. The minimum number of colors needed to produce a proper coloring of a graph $G$ is called a *chromatic number* of $G$ and denoted by $\chi(G)$ [2]. If $\chi(G) = k$ means that the vertices of a graph G can be colored by k color, but it cannot be color by $k - 1$ color.

**Theorem 1.1** [1] A graph $G$ with orders $n$ has chromatic numbers equal to $n$ if and only if $G$ is a complete graph, namely $G = K_n$. ■

**Theorem 1.2** [1] If $H$ is a subgraph of a graph $G$, then $\chi(H) \leq \chi(G)$. ■

A chromatic polynomial of a graph was first introduced by George David Birkhof in 1912 and continued by Whitney in 1932. A chromatic polynomial of a graph $G$, denoted by $P(G, k)$, is a polynomial which encodes the number of distinct ways to color the vertices of $G$ with $k$ colors (where colorings are counted as distinct even if they differ only by permutation of colors). The chromatic number $\chi(G)$ is the least natural number $k$ for which such a partition is possible. If $k < \chi(G)$ then $P(G, k) = 0$. In this paper, we examined the chromatic polynomial of a fan graph.

The chromatic polynomial of some graphs have been obtained. A graph having n vertices but 0 edge, $N_n$, have the chromatic polynomial $P(N_n, k) = k^n$, while a complete graph on $n$ vertices, $K_n$, have the chromatic polynomial $P(K_n, k) = k\,(k-1)(k-2)\dots(k-$

$n+1$) . Read [4] proved that the chromatic polynomial of any tree having $n$ vertices, $T_n$, is $P(T_n, k) = k\ (k-1)^{n-1}$. A *color-partition* of a graph $G = (V,E)$ is a partition of $V$ into disjoint non-empty subsets, $V = V_1 \cup V_2 \cup \cdot \cdot \cdot \cup V_k$, such that the color-class $V_i$ is an independent set of vertices in $G$, for each $1 \leq i \leq k$.

**Theorem 1.3** [3] Let $G$ be a graph of order $n$. Then, the chromatic polynomial of a graph $G$ is $P(G,k) = \sum_{i=1}^{n} \alpha(G,i)(k)_i$ where $\alpha(G,i)$ is the number of color-partitions of $G$ into $i$ color-classes. ∎

Read [4] gave the properties of the chromatic polynomial of a graph G with $n$ vertices and $m$ edges, in the following theorem.

**Theorem 1.4** [4] Let $P(G,k) = a_n k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0$ be a chromatic polynomial of a graph $G$ with $n$ vertices and $m$ edges, then the following conditions are satisfied.

a. All the coefficients are integers (could be 0).
b. The order of the polynomial is $n$.
c. The coefficients of $k^n$ is 1: $a_n = 1$.
d. The coefficients of $k^{n-1}$ is – $m$: $a_{n-1} = -m$.
e. The coefficients of $k^0$ is 0: $a_0 = 0$.
f. Signs of coefficients alternate between positive and negative
g. If $m \neq 0$, then the sum of the coefficients on $P(G,k)$ is 0. ∎

## 2. Main Results

A fan graph $F_n$ is a simple graph formed by connecting a single vertex to all vertices of a path on $n$ vertices $P_n$. So, the fan graph $F_n$ has $n+1$ vertices and $2n-1$ edges. Let $V(F_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ be the vertex set of $F_n$, where $deg(v_0) = n$, $deg(v_1) = deg(v_n) = 2$ and $deg(v_i) = 3$ for $2 \leq i \leq n-1$ and $E(F_n) = \{v_0 v_i | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\}$ be the edge set of $F_n$. Before we discuss the chromatic polynomial of a fan, we give the chromatic number of a fan.

**Lemma 2.1** Let $n \geq 3$ be an integer. Then, the chromatic number of a fan $F_n$ is 3, namely $\chi(F_n) = 3$.

**Proof**. Since $F_n$ contain a triangle $K_3$ as a subgraph, according to Theorem 1.1 and Theorem 1.2, then $\chi(F_n) \geq 3$. Next, we will show that $\chi(G) \leq 3$ by giving color to fan graph. We define a coloring of the vertices of fan graph $F_n$ as follows.

$$f(v_i) = \begin{cases} 1 & ; \quad i = 0 \\ 2 & ; \quad i \text{ odd} \\ 3 & ; \quad i \text{ even and } i \neq 0 \end{cases}$$

It can be seen easily that each adjacent vertex has a different color. So, $\chi(G) \leq 3$. ■

Now, we determine the chromatic polynomial of fan graph $F_n$ for each integer $n \geq 3$. We consider a fan $F_3$ as depicted in Figure 1.

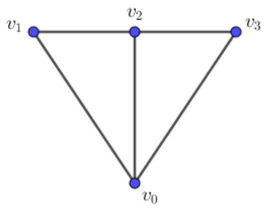


Figure 1. A fan $F_3$

First, we look for all possible coloring of all vertices of $F_3$. Since $\chi(F_3) = 3$, all possible coloring in all vertices of $F_3$ started with three colors up to the number of vertices, that is four colors. Next, we determine the number of color-partitions of a fan $F_3$. Table 1 shows that all possibilities of coloring and partitioning of the vertex set of a fan $F_3$.

Table 1. A coloring possibility of $F_3$

| If $v_i$ and $v_j$ have the same color | Another possible coloring | The color-classes | Number of partitions |
|---|---|---|---|
| $v_1 = v_3$ | $v_0 \neq v_2$ | $\{v_1, v_3\}, \{v_2\}, \{v_0\}$ | 3 |
| All distinct | $v_0 \neq v_1 \neq v_2 \neq v_3$ | $\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}$ | 4 |

Based on Table 1, the number of color-partitions of $F_3$ into *i* color-classes are $\alpha(F_3, 3) = 1$ and $\alpha(F_3, 4) = 1$. Thus, by Theorem 1.3 we obtain

$$\begin{aligned} P(F_3, k) &= \sum_{i=1}^{4} \alpha(F_3, i)(k)_i \\ &= 0k_1 + 0k_2 + k_3 + k_4 \\ &= k(k-1)(k-2) + k(k-1)(k-2)(k-3) \\ &= k(k-1)(k-2)(1 + (k-3)) \\ &= k(k-1)(k-2)(k-2) \\ &= k(k-1)(k-2)^2 \end{aligned}$$

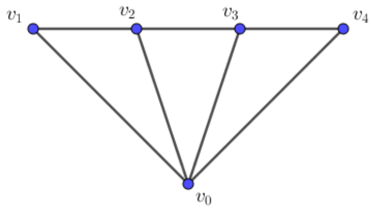


Figure 2. A fan $F_4$

Now, we consider a fan graph $F_4$ as depicted in Figure 2. Table 2 shows that all possibilities of coloring and partitioning of the vertex set of a fan $F_4$.

Table 2. A coloring possibility of $F_4$

| If $v_i$ and $v_j$ have the same color | Another possible coloring | The color-classes | Number of partitions |
|---|---|---|---|
| $v_1 = v_3$ | $v_2 = v_4 \neq v_0$ | $\{v_1, v_3\}, \{v_2, v_4\}, \{v_0\}$ | 3 |
| $v_1 = v_3$ | $v_2 \neq v_4 \neq v_0$ | $\{v_1, v_3\}, \{v_2\}, \{v_4\}, \{v_0\}$ | 4 |
| $v_1 = v_4$ | $v_2 \neq v_3 \neq v_0$ | $\{v_1, v_4\}, \{v_2\}, \{v_3\}, \{v_0\}$ | 4 |
| $v_2 = v_4$ | $v_1 \neq v_3 \neq v_0$ | $\{v_1, v_4\}, \{v_1\}, \{v_3\}, \{v_0\}$ | 4 |
| All distinct | $v_0 \neq v_1 \neq v_2 \neq v_3 \neq v_4$ | $\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$ | 5 |

Based on Table 2, the number of color-partitions of $F_4$ into *i* color-classes are $\alpha(F_4, 3) = 1$, $\alpha(F_4, 4) = 3$ and $\alpha(F_4, 5) = 1$. So, by Theorem 1.3 we obtain the chromatic polynomial of a graph $F_4$ as follows.

$$
\begin{aligned}
P(F_3, k) &= \sum_{i=1}^{4} \alpha(F_3, i)(k)_i \\
&= 0k_1 + 0k_2 + k_3 + 3k_4 + k_5 \\
&= k(k-1)(k-2) + 3\big(k(k-1)(k-2)(k-3)\big) + k(k-1)(k-2)(k-3)(k-4) \\
&= k(k-1)(k-2)(1 + 3(k-3) + (k-3)(k-4)) \\
&= k(k-1)(k-2)(1 + 3k - 9 + k^2 - 7k + 12) \\
&= k(k-1)(k-2)(k^2 - 4k + 4) \\
&= k(k-1)(k-2)(k-2)^2 \\
&= k(k-1)(k-2)^3
\end{aligned}
$$

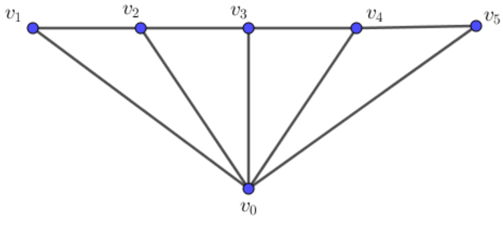


Figure 3. A fan $F_5$

Furthermore, we consider a fan graph $F_5$ as depicted in Figure 3. The coloring of all vertices of $F_6$ starts with 3 colors up to the number of vertices which is 6 colors. Table 3 shows that all possibilities of coloring and partitioning of the vertex set of a fan $F_5$.

Table 3. A coloring possibility of $F_5$

| If $v_i$ and $v_j$ have the same color | Another possible coloring | The color-classes | Number of partitions |
|---|---|---|---|
| $v_2 = v_4$ | $v_1 = v_3 = v_5 \neq v_0$ | $\{v_2, v_4\}, \{v_1, v_{3,} v_5\}, \{v_0\}$ | 3 |
| $v_1 = v_3$ | $v_2 = v_4 \neq v_5 \neq v_0$ | $\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}, \{v_0\}$ | 4 |
| $v_1 = v_3$ | $v_2 = v_5 \neq v_4 \neq v_0$ | $\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}, \{v_0\}$ | 4 |
| $v_1 = v_4$ | $v_2 = v_5 \neq v_3 \neq v_0$ | $\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}, \{v_0\}$ | 4 |
| $v_1 = v_4$ | $v_3 = v_5 \neq v_2 \neq v_0$ | $\{v_1, v_4\}, \{v_3, v_5\}, \{v_2\}, \{v_0\}$ | 4 |
| $v_1 = v_5$ | $v_2 = v_4 \neq v_3 \neq v_0$ | $\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}, \{v_0\}$ | 4 |
| $v_2 = v_4$ | $v_3 = v_5 \neq v_1 \neq v_0$ | $\{v_2, v_4\}, \{v_3, v_5\}, \{v_1\}, \{v_0\}$ | 4 |
| $v_1 = v_3 = v_5$ | $v_2 \neq v_4 \neq v_0$ | $\{v_1, v_3, v_5\}, \{v_2\}, \{v_4\}, \{v_0\}$ | 4 |
| $v_1 = v_3$ | $v_2 \neq v_4 \neq v_5 \neq v_0$ | $\{v_1, v_3\}, \{v_2\}, \{v_4\}, \{v_5\}, \{v_0\}$ | 5 |
| $v_1 = v_4$ | $v_2 \neq v_3 \neq v_5 \neq v_0$ | $\{v_1, v_4\}, \{v_2\}, \{v_3\}, \{v_5\}, \{v_0\}$ | 5 |
| $v_1 = v_5$ | $v_2 \neq v_3 \neq v_4 \neq v_0$ | $\{v_1, v_5\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_0\}$ | 5 |
| $v_2 = v_4$ | $v_1 \neq v_3 \neq v_5 \neq v_0$ | $\{v_2, v_4\}, \{v_1\}, \{v_3\}, \{v_5\}, \{v_0\}$ | 5 |
| $v_2 = v_5$ | $v_1 \neq v_3 \neq v_4 \neq v_0$ | $\{v_2, v_5\}, \{v_1\}, \{v_3\}, \{v_4\}, \{v_0\}$ | 5 |
| $v_3 = v_5$ | $v_1 \neq v_2 \neq v_4 \neq v_0$ | $\{v_3, v_5\}, \{v_1\}, \{v_2\}, \{v_4\}, \{v_0\}$ | 5 |
| All distinct | $v_0 \neq v_1 \neq v_2 \neq v_3 \neq v_4 \neq v_5$ | $\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$ | 6 |

Based on Table 3, the number of color-partitions of $F_5$ into *i* color-classes are $\alpha(F_5, 3) = 1$, $\alpha(F_5, 4) = 7$, $\alpha(F_5, 5) = 6$ and $\alpha(F_5, 6) = 1$. So, by Theorem 1.3 we obtain the chromatic polynomial of a graph $F_5$ as follows.

$$
\begin{aligned}
P(F_5, k) &= \sum_{i=1}^{6} \alpha(F_5, i)(k)_i \\
&= 0k_1 + 0k_2 + k_3 + 7k_4 + 6k_5 + k_6 \\
&= k(k-1)(k-2) + 7\big(k(k-1)(k-2)(k-3)\big) + 6\big(k(k-1)(k-2)(k-3)(k-4)\big) + k(k-1)(k-2)(k-3)(k-4)(k-5) \\
&= k(k-1)(k-2)[1 + 7(k-3) + 6(k-3)(k-4) + (k-3)(k-4)(k-5)] \\
&= k(k-1)(k-2)[1 + 7k - 21 + 6k^2 - 42k + 72 + k^3 - 12k^2 + 47k - 60] \\
&= k(k-1)(k-2)[k^3 - 6k^2 + 12k - 8] \\
&= k(k-1)(k-2)(k-2)^3 \\
&= k(k-1)(k-2)^4.
\end{aligned}
$$

In general, we have that the chromatic polynomial of a fan $F_n$ for each integer $n \geq 3$ is $P(F_n, k) = k(k-1)(k-2)^{n-1}$. So, we have the following theorem.

**Theorem 2.1** Let $n \geq 3$ be a positive integer. The chromatic polynomial of a fan graph $F_n$ is

$$P(F_n, k) = k(k-1)(k-2)^{n-1}.$$

**Proof.** A fan graph $F_n$ has $n+1$ vertices, where one vertex, called a center, of degree $n$, two vertices of degree 2, and the others of degree 3. The center can be colored with $k$ colors, the first vertex of degree 2 can be colored with $k-1$ colors and each one of the other $n-1$ vertices, in order, can be colored with $k-2$ colors. Therefore,

$$P(F_n, k) = k(k-1)(k-2)^{n-1}. \quad \blacksquare$$

For example, there are 6 different ways to color a fan $F_3$ with 3 colors, red, blue, and green, as depicted in Figure 4.

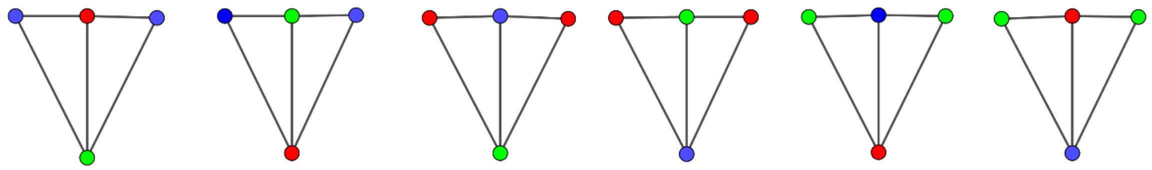

Figure 4. Six different ways to color a fan $F_3$ with 3 colors

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