On The Existence of Non-Diregular Digraphs of Order Two Less than the Moore Bound*

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ABSTRACT

A communication network can be modelled as a graph or a directed graph, where each processing element is represented by a vertex and the connection between two processing elements is represented by an edge (or, in case of directed connections, by an arc). When designing a communication network, there are several criteria to be considered. For example, we can require an overall balance of the system. Given that all the processing elements have the same status, the flow of information and exchange of data between processing elements will be on average faster if there is a similar number of interconnections coming in and going out of each processing element, that is, if there is a balance (or regularity) in the network. This means that the in-degree and out-degree of each vertex in a directed graph (digraph) must be regular. In this paper, we present the existence of digraphs which are not diregular (regular out-degree, but not regular in-degree) with the number of vertices two less than the unobtainable upper bound for most values of out-degree and diameter, the so-called Moore bound.

Keywords : Non-diregular digraphs, Moore bound

INTRODUCTION

Large communication network design has become a growing interest due to recent advances in very large scale integrated technology. In such networks, it is desirable to have connections which achieve the most efficient and reliable communication in view of practical economic constraint.

In communication network design, there are several factors which should be considered. Two of the factors seem to appear most frequently, namely, (a) the number of connections which can be attached to a processing element is limited, and (b) a short communication route between any two processing elements is required. We would like to end up with a large network subject to these constraints.

Another factor that may be considered when designing a communication network is fault tolerance. The fault tolerance of a communication network is the capability of the network to continue working when a number of processing elements or connections have failed. The larger number of faulty processing elements or connections that can be tolerated the better fault tolerance.

When designing a communication network, we may require an overall balance of the system. Given that all the processing elements have the same status, the flow of information and exchange of data between processing elements will be on average faster if there is a similar number of interconnections coming in and going out of each processing element, that is, if there is a balance (or regularity) in the network.

The underlying topology of a network is usually studied by graph-theoretic means: a network is represented by a graph which consists of a set of vertices; some or all pairs of the vertices may be joined by edges (or, in case of directed connections, by arcs). Each processor is represented by a vertex and two particular vertices are joined by an edge (respectively, arc) if there is a two-way (respectively, one-way) connection between the two corresponding processors.

The number of connections incident to vertex A is called the it *degree* of *A*; if the connections are one way only then we make a distinction between in-coming and out-going connections and speak of the *in-degree* and the *out-degree* of *A*. If every vertex has the same in-degree (respectively, out-degree) then the network is said to be *in-regular* (respectively, *out-regular*). If the network (digraph) is in-regular and out-regular, then it is called *diregular*. The *distance* from vertex *A* to vertex *B* is the length of the shortest path from *A* to *B* measured by the number of edges (or arcs) that need to be traversed in order to reach B from A. The maximum distance between any pair of vertices is called the *diameter* of the graph.

Among the many factors to be considered in a network design, there are two which seem to appear most frequently: The number of connections that can be attached to a vertex is limited, and a short communication route between any two vertices is required. Naturally, one would like to end up with a large network subject to these constraints. In graph-theoretic terms this corresponds to the following well-known fundamental problem.

Degree/diameter problem
Construct graphs of given maximum degree d and diameter k with the largest possible number of vertices.

The directed version of the problem differs only in that `degree' is replaced by `out-degree' in the statement of the problem. We treat separately the case of directed networks because although the extremal problems arising there are superficially similar to the undirected case, they do require substantially different techniques.

A straightforward general upper bound on the largest possible order (i.e. number of vertices) $n_{d,k}$ of a graph of maximum degree d and diameter k is the *Moore bound* $M_{d,k}$, named after E. F. Moore who first proposed the problem:

$n_{d,k} \leq M_{d,k} = 1 + d + d(d - 1) + \ldots + d(d - 1)^{k-1}$

for undirected graphs, and

$n_{d,k} \leq M_{d,k} = 1 + d + d^2 + \ldots + d^k$

for directed graphs (digraphs).

In general, these bounds cannot be attained. For the case of directed graphs, the bound can be obtained only when $d = 1$ or $k = 1$ (Bridges & Toueg 1980). Since there is no digraph whose order attained the Moore bound for maximum out-degree $d \geq 2$ and diameter $k \geq 2$, the research of the existence of large digraphs focuses on digraphs whose order is close to the Moore bound.

The research on this area has been carried out by several world class researchers. Most of the research activities focus on the two foll.owing mainstreams:
proving the non-existence of digraphs of order `close' to the Moore bound, for example (Baskoro *et al*. 2005, Baskoro *et al*. 1998, Hoffman & Singleton 1960, Miller & Siran 2001) constructing large digraphs, for example (Comellas & Fiol 1990, Fiol *et al*. 1984, Miller & Slamin 2000).

This motivated author to conduct another research activity concerning digraphs with order close to the Moore bound, that is, the study of the regularity of such digraphs. This paper deals mainly with the issue of diregularity of directed graphs (digraphs) with the number of vertices close to the unobtainable upper bound for most values of out-degree d and diameter k.

PREVIOUS RESULTS

The study of diregularity of a digraph is conducted by proving in-regularity of digraph as well as its out-regularity. If such digraph is both in-regular and out-regular, then the digraph is diregular.

The digraph whose order is equal to the Moore bound is called *Moore digraph*. Moore digraph has diameter equal to k and the out-degree of each of its vertices equal to d. It is easy to show that all in-degrees in such a Moore digraph must be equal to d. These observations follow from the well known classification (Bridges & Toueg S 1980; Plesnik & Znam 1974) which says that Moore digraphs exist only in the trivial cases when d = 1 (directed cycles of length $k + 1$ for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d + 1$ for any $d \geq 1$).

The research continued on the diregularity of digraph whose order is close to the Moore bound ($M_{d,k} - \delta$, for $1 \leq \delta < d$). The out-regularity of digraphs of order close to the Moore bound was observed in (Baskoro *et al*. 1998). We state the observation in the following theorem.

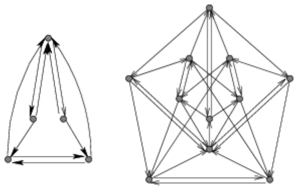
Theorem 1 [Baskoro, Miller and Plesník] *The digraph of maximum out-degree $d \geq 2$, diameter $k \geq 2$ and order $M_{d,k} - \delta$, for $1 \leq \delta < d$, must be out-regular of out-degree d.*

Proof. Suppose that the digraph contains a vertex u with out-degree $d_1 < d$ (i.e., $d_1 \leq d$-1), then considering the number of vertices in the out-bound spanning tree starting from vertex u, the order of the digraph:

$$n \le 1 + d_1 + d_1 d + ... + d_1 d^{k-1}$$
$$n = 1 + d_1(1 + d + ... + d^{k-1})$$
$$n \le 1 + (d-1)(1 + d + ... + d^{k-1})$$
$$n = (1 + d + ... + d^k) - (1 + d + ... + d^{k-1})$$
$$n < M_{d,k} - d$$

Hence the out-degree of any vertex in a digraph of order $M_{d,k} - \delta$, for $1 \le \delta < d$, must be equal to d, that is, such a digraph must be out-regular.

In contrast, establishing the in-regularity of digraphs of order close to the Moore bound is not as easy as proving the out-regularity of digraphs. Figure 1 shows the fact that there *exist* a digraph of out-degree $d = 2$, diameter $k = 2$ and order $M_{2,2} - 2$ as well as a digraph of out-degree $d = 3$, diameter $k = 2$ and order $M_{3,2} - 3$ in which *not all* vertices have the same in-degree.



G_2 G_3

Figure 1. Non-diregular digraphs G_2 and G_3.

Some results on diregularity of digraph whose order is close to the Moore bound ($n = M_{d,k} - \delta$ where $1 \le \delta < d$) have been established. Miller, Gimbert, Siran and Slamin (2000) proved that every digraph whose order is one less than Moore bound (n = $n = M_{d,k} - 1$) is diregular (Miller *et al*. 2000).

For the case of digraph whose order is two less than Moore bound ($n = M_{d,k} - 2$), Slamin and Miller have proved that the digraphs of out-degree $d = 2$, diameter $k > 2$ and order $n = M_{d,k} - 2$ are diregular (Slamin & Miller M 2000). For out-degree $d = 3$, Slamin, Baskoro & Miller (2001) have investigated some properties and structures of the diregularity of digraphs of order two less than Moore bound, however, it is not known whether the such digraphs are diregular or not.

In this paper, we present the existence of non-diregular digraphs of out-degree $d \ge 2$, diameter $k = 2$ and order two less than Moore bound ($n = M_{d,2} - 2$) as described in the following section.

RESEARCH PROCEDURE

This research is conducted by investigating the known results on the construction techniques of large digraphs that have been established, especially vertex deletion scheme (Miller & Slamin 2000). In detail, the procedure of the research is as follows.

1. Observing the largest known digraphs given out-degree degree $d \ge 2$, diameter $k \ge 2$ and order $n = d^k + d^{k-1}$
2. Observing the construction techniques of large digraphs using vertex deletion scheme.
3. Applying vertex deletion scheme technique to the largest known digraphs.
4. Investigating the order of generated digraphs obtained from vertex deletion scheme.

Investigating the diregularity of generated digraphs obtained from vertex deletion scheme.

MAIN RESULTS

Before presenting the results, we present the definition of Kautz digraph and theorem on construction technique of digraph so-called *vertex deletion scheme.*

Definition 1. *Kautz digraph, denoted by K(d,k), is the diregular digraph of out-degree d* ≥ 2, *diameter* $k \ge 2$ *and order* $n = d^k + d^{k-1}$

If the diameter of Kautz digraph is $k = 2$, then the order is $n = d^2 + d = M_{d,2} - 1$ for $d \ge 2$. Kautz digraph can be obtained from complete digraph by using *line digraph* construction technique. This construction technique can be described as follows. Let $G = (V,A)$ and let N be the set of all walks of length 2 in G. The line digraph of a digraph G, $L(G) = (A,N)$, is a digraph such that the set of vertices of $L(G)$ is equal to the set of arcs of G and the set of arcs of $L(G)$ is equal to the set of walks of length 2 in G. This means that a vertex uv of $L(G)$ is adjacent to a vertex wx if and only if $v = w$.The line digraph construction technique can be used to construct a new digraph whose order is larger than the original digraph. This construction technique provides the diregularity on the degree of the original digraph which means that if digraph G is diregular of degree d then the line digraph $L(G)$ is also diregular of degree d. Figure 2 shows the complete digraph K_5 and Kautz digraph $Ka(4,2)$ of degree 4, diameter 2 and order $n = 4^2 + 4 = 20$ which is obtained from K_5 using line digraph construction technique $L(K_5)$.

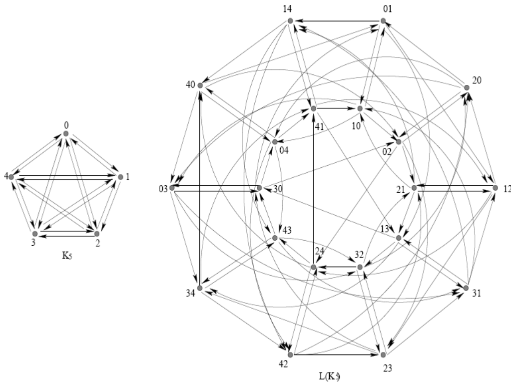


Figure 2. Complete digraph K_5 and its line digraph $Ka(4,2)$.

Teorema 2. [Miller and Slamin]. If G is the digraph of out-degree d, diameter k and order n with property that out-neighbours of vertex u are equal to the out-neighbours of vertex v, then there is a digraph G' of out-degree d, diameter at most k and order n – 1.

Digraph G' is constructed from G by utilizing vertex deletion scheme. The procedure of this technique can be described as follows. Let G be the digraph of out-degree d, diameter k and order n. Let $N^+(v)$ denotes the set of out-neighbour vertices of vertex v. If u and v are vertices in G with property that $N^+(u) = N^+(v)$, then G' can be obtained by deleting vertex u together with all arcs going out from u and reconnecting all incoming arcs of vertex u to vertex v.

Digraphs which are obtained from the line digraph iterations always contain some pairs of vertices with the same out-neighbourhoods. Since Kautz digraph Ka(d,k) of out-degree d, diameter k and order $d^k + d^{k-1}$ can be constructed using the k-1 iterations line digraph of complete digraph K_{d+1}, we can apply Theorem 2 to obtain new digraphs with order less than Kautz digraphs.

In the following theorem we shall prove that the new digraphs obtained from Kautz digraph using vertex deletion scheme are not diregular.

Teorema 3. There exist non-diregular digraphs of out-degree $d \geq 2$, diameter k = 2, and order two less than Moore bound ($n = M_{d,2} - 2$).

Proof. Let Ka(d,2) be the Kautz digraph of out-degree $d \geq 2$ and diameter k = 2. By definition of Kautz digraph, the order of Ka(d,2) is $n = d^2 + d$, that is one less than Moore bound or $n = M_{d,2} - 1$. Since Ka(d,2) has d + 1 distinct sets containing d vertices with the same out-neighbourhoods, so we can choose two vertices, say u and v with property that $N^+(u) = N^+(v)$. By Theorem 2, we can construct new digraph G', by deleting vertex u together with all arcs going out from u and reconnecting all incoming arcs of vertex u to vertex v. The new digraph G' has out-degree d, diameter at most k and order $n - 1 = M_{d,2} - 1 - 1 = M_{d,2} - 2$. We know that Ka(d,2) is diregular. Thus all arcs which are previously going to u and now are going to vertex v implies that the in-degree of v is larger than d. Consequently, digraph G' is not in-regular which means that digraph G' is not diregular.

Figure 3 illustrates the digraph of out-degree 4, diameter 2 and order 19 that is constructed from Kautz digraph Ka(4,2) as shown in Figure 2 by deleting vertex 12 and reconnecting its incoming arcs to vertex 02.

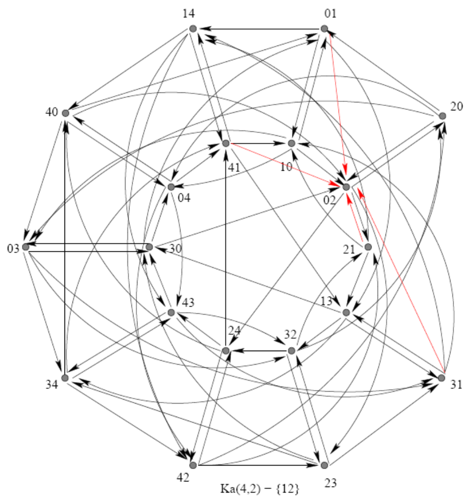


Figure 3. Non diregular digraph of order 19.

CONCLUSION

We conclude this paper with summary of known results on the diregularity of digraphs which is updated from (Comellas & Fiol 1990) as presented in Table 1 and research direction for future works.

Tabel 1. Diregularity of digraph out-degree *d*, diameter *k* and order *n*.

d	k	n	Diregularity
1	≥ 1	$M_{1,k}$	Only diregular
≥ 1	1	$M_{d,1}$	Only diregular
≥ 2	≥ 2	$M_{d,k}$-1	Only diregular
2,3	2	$M_{d,2}$-2	Diregular & non diregular
2	≥ 3	$M_{2,k}$-2	Only diregular
≥ 4	2	$M_{d,2}$-2	Non diregular (the result), diregular (unkown)
≥ 3	≥ 3	$M_{d,k}$-2	Not kwon

Table 1 shows that the diregularity of digraphs whose order are two less than the Moore bound ($n = M_{d,k} - 2$) are still unknown, in particular for out-degree $d \geq 3$ and diameter $k \geq 3$.

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