On $\tau[M]$-Cohereditary Modules

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ABSTRACT

Let R be a ring with unity and N a left R-module. Then N is said linearly independent to R (or N is R-linearly independent) if there exists a monomorphism $\varphi : R^{(\Lambda)} \to N$. We can define a generalization of linearly independency relative to an R-module M. N is called M-linearly independent if there exists a monomorphism $\varphi : M^{(\Lambda)} \to N$. A module Q is called M-sublinearly independent if Q is a factor module of a module which is M-linearly independent. The set of M-sublinearly independent modules is denoted by $\tau[M]$. It is easy to see that $\tau[M]$ is subcategory of category R-Mod. Furthermore, any submodule, factor module and external direct sum of module in $\tau[M]$ are also in $\tau[M]$. A module is called $\tau[M]$-injective if it is P-injective, for all modules P in $\tau[M]$. Q is called $\tau[M]$-cohereditary if Q $\in \tau[M]$ and any factor module of Q is $\tau[M]$-injective. In this paper, we study the characterization of category $\tau[M]$-cohereditary modules. For any Q in $\tau[M]$, Q is a $\tau[M]$-cohereditary if and only if every submodule of Q-projective module in $\tau[M]$ is Q-projective. Moreover, Q is a $\tau[M]$-cohereditary if and only if every factor module of Q is a direct summand of module which contains this factor module. Also, we obtain some cohereditary properties of category $\tau[M]$. There are: for any R-modules P, Q. If Q is P-injective and every submodule of P is Q-projective, then Q is cohereditary (1); if P is Q-projective and Q is cohereditary, then every submodule of P is Q-projective (2); a direct product of modules which $\tau[M]$-cohereditary is $\tau[M]$-cohereditary (3). The cohereditary characterization and properties of category $\tau[M]$ above is truly dual of characterization and properties of category $\tau[M]$.

Keywords : Category $\tau[M]$, Q-projective, P-injective, $\tau[M]$-cohereditary

INTRODUCTION

We denote *R* as a associative ring with unity $1 \neq 0$, and by module we mean a left *R*-module. Let *N* be a left *R*-module, then *N* is said linearly independent to *R* (or *N* is *R*-linearly independent) if there exists a monomorphism $\varphi : R^{(\Lambda)} \to N$. We can define a generalization of linearly independency relative to an *R*-module *M*. *N* is called *M*-linearly independent if there exists a monomorphism $\varphi : M^{(\Lambda)} \to N$.

The set of factor module of module which *M*-linearly independen is denoted by $\tau[M]$ It is easy to see that $\tau[M]$ is subcategory of category *R*-Mod. Furthermore, any submodule, factor module and external direct sum of module in $\tau[M]$ are also in $\tau[M]$

A module *P* is called $\sigma[M]$-projective if it is *Q*-projective, for all modules *Q* in $\sigma[M]$. A module *P* is called $\sigma[M]$-hereditary if *P* $\in \sigma[M]$ and any it is submodule is $\sigma[M]$-projective. More over about $\sigma[M]$-projective and $\sigma[M]$-hereditary, see (Wisbauer 1991).

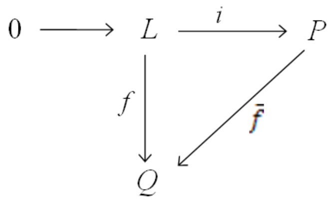
In (Garminia 2006), showed that for any M is projective module and Q is $\sigma[M]$-injective in $\sigma[M]$, then Q is $\sigma[M]$-cohereditary module if and only if every submodule of Q-projective module in $\sigma[M]$ is Q-projective.

A module Q is called $\tau[M]$-injective if it is P-injective, for all modules P in $\tau[M]$. A module Q is called $\tau[M]$-cohereditary if $Q \in \tau[M]$ and any of its factor module is $\tau[M]$-injective. In this paper, we showed similar in (Garminia 2006) but not necessary that M must be projective module.

RESULTS AND DISCUSSION

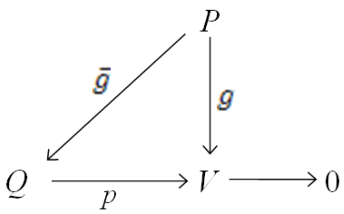
$\tau[M]$-Injective Modules

Definition 2.1. Let P, Q be left R-modules. A module Q is said P-injective if for any R-homomorphism $f : L \to Q$ defined on a submodule L of P can be extended to an R-homomorphism $\overline{f} : P \to Q$ with $f = \overline{f}i$, where $i : L \to P$ is the natural inclusion mapping, or we consider the following diagram is commute:



A module is called $\tau[M]$-injective if it is injective for all module in $\tau[M]$.

Definition 2.2. Let P, Q be left R-modules. A module P is said Q-projective if for any R-homomorphism $g : V \to P$ defined on a factor module V of Q can be extended to an R-homomorphism $\overline{g} : Q \to P$ with $g = p\overline{g}$, where $p : Q \to V$ is the natural mapping, or we consider the following diagram is commute:



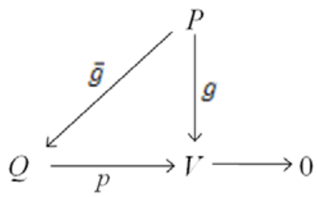
A module is called $\tau[M]$-projective if it is projective for all module in $\tau[M]$.

It can be seen that injectivity and projectivity are dual notions. Now we give the basic properties of relationship between injective and projective modules.

Lemma 2.3. ***L****et P, Q be R-modules in* $\tau[M]$.

i. *If Q is* $\tau[M]$*-injective and every submodule of P is a Q-projective for any Q in* $\tau[M]$*, then every factor module of Q is a* $\tau[M]$*-injective.*
ii. *If P is a* $\tau[M]$*-projective and every factor module of Q is a P-injective, then every submodule of P is a* $\tau[M]$*-projective.*

Proof: (i) Let L be a submodule of P in $\tau[M]$ and let V a factor module of Q. Suppose f is an R-homomorphism from $L \to V$. Consider the following diagram:



Since L is a Q-projective, then there exists R-homomorphism $\overline{f} : L \to Q$ which satisfies:

$$f = p(\overline{f}) \qquad (1)$$

Also, since Q is a P-injective, then there exists R-homomorphism $\overline{\overline{f}} : P \to Q$ which satisfies:

$$\overline{f} = \overline{\overline{f}}i \qquad (2)$$

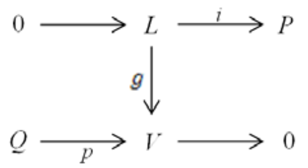
By (1) and (2), we have:

$$f = p\overline{f} = p\left(\overline{\overline{f}}i\right) = \left(p\overline{\overline{f}}\right)i.$$

It means, for any R-homomorphism $f : L \to V$, there exists R-homomorphism $p\overline{\overline{f}} : P \to V$ which satisfies:

$$f = \left(p\overline{\overline{f}}\right)i \ .$$

This implies V is P-injective for any P in $\tau[M]$. So, V is $\tau[M]$-injective for every factor module V of Q.
(ii) Let L be a submodule of P and let V a factor module of Q in $\tau[M]$. Suppose g is an R-homomorphism from $L \to V$. Consider the following diagram:



Since V is P-injective, then there exists R-homomorphism $\overline{g} : P \to V$ which satisfies:

$$g = \overline{g}i \qquad (3)$$

Also, since P is a Q-projective, then there exists R-homomorphism $\overline{\overline{g}} : P \to Q$ which satisfies:

$$\overline{g} = p\overline{\overline{g}} \qquad (4)$$

By (3) and (4), we have:

$$g = \overline{g}i = \left(p\overline{\overline{g}}\right)i = p\left(\overline{\overline{g}}i\right).$$

It means, for any R-homomorphism $g : L \to V$, there exists R-homomorphism $\overline{\overline{g}}i : L \to Q$ Q which satisfy:

$$g = p\left(\overline{\overline{g}}i\right).$$

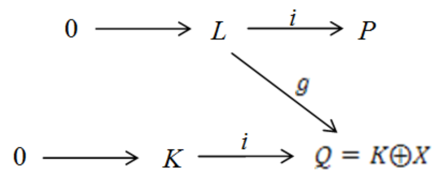
This implies L is Q-projective for any Q in $\tau[M]$. So L is $\tau[M]$-projective for every submodule L of P. □

Now that we know that injective modules are plentiful, it is worthwhile to list some of their basic properties. Suppose that $\{Q_i \mid i \in I\}$ is a family of R-modules. Then we let $Q = \prod_{i\in I} Q_i$ denote their strong, or complete, direct sum. Thus the elements of Q are I-tuples $\prod_i Q_i$ with $q_i \in Q_i$ and with no assumption on the number of nonzero components. If I is finite, then $\prod_{i\in I} Q_i = \oplus\sum_i Q_i$. Note that $\prod_i Q_i$ is also called the direct product of Q_i.

The following proposition is injectivity property of direct summand of modules in category $\tau[M]$.

Proposition 2.4. *Let Q be an R-module. If Q is $\tau[M]$-injective and K is a direct summand of Q, then K is $\tau[M]$-injective.*

Proof: Let L be a submodule of P with P in $\tau[M]$ and let K be a direct summand of Q. Suppose g is an R-homomorphism from $L \to Q$. Consider the following diagram:



Since Q is P-injective for any P in $\tau[M]$, then there exists R-homomorphism $\overline{g} : P \to Q$ Q which satisfies:

$$g = \overline{g}i$$

Also, since K is a direct summand of Q, then there exists an R-homomorphism $\pi : Q \to K$ such that $\pi i = I_K$. This implies, for any R-homomorphism $\pi g : L \to K$, there exists an R-homo-morphism $\pi\overline{g} : P \to K$ which satisfy:

$$\pi g = \left(\pi\overline{g}\right)i.$$

It means K is P-injective for any P in $\tau[M]$. So K is $\tau[M]$-injective. □

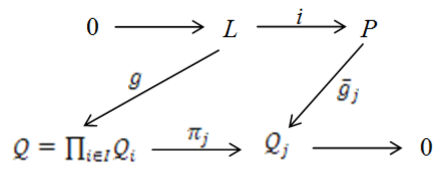
Proposition 2.5. *Let Q_i be an R-module and let $\{Q_i \mid i \in I\}$ be a family of injective module. Then the following assertions are equivalent:*

i. $\{Q_i \mid i \in I\}$ *is* $\tau[M]$*-injective.*

ii. $Q = \prod_{i\in I} Q_i$ *is* $\tau[M]$*-injective.*

Proof: $(i \Rightarrow ii)$ Let L be a submodule of P with P in $\tau[M]$ and let $Q = \prod_{i\in I} Q_i$ be a direct product of Q_i. Suppose g is an R-homomorphism from $L \to Q$. Since Q_j is P-injective for any P in $\tau[M]$ and $j \in I$, then for any R-homomorphism $\pi_j g : L \to Q_j$, there exists an R-homomorphism $\overline{g}_j : P \to Q_j$ which satisfy:

$$\pi_j g = \overline{g}_j i \qquad (5)$$



Hence we obtain product an R-homomorphism $\overline{g} : P \to Q = \prod_{i\in I} Q_i$ with

$$\overline{g}_j = \pi_j \overline{g}$$

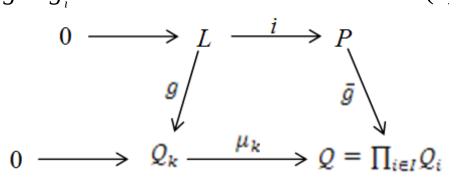
and we have (5), which implies:

$$g = \overline{g} i \ .$$

It means $Q = \prod_{i\in I} Q_i$ is P-injective for any P in $\tau[M]$. So $Q = \prod_{i\in I} Q_i$ is $\tau[M]$-injective.

$(i \Leftarrow ii)$ Let L be a submodule of P with P in $\tau[M]$ and let $Q = \prod_{i\in I} Q_i$ be a direct product of Q_i. Suppose g is an R-homomorphism from $L \to Q_k$ for any $k \in I$. Since $Q = \prod_{i\in I} Q_i$ is P-injective for any P in $\tau[M]$, then for any R-homomorphism $\mu_k g : L \to Q = \prod_{i\in I} Q_i$, there exists an R-homomorphism $\overline{g} : P \to Q = \prod_{i\in I} Q_i$ which satisfy:

$$\mu_k g = \overline{g}_i \qquad (6)$$



Hence we obtain product an R-homomorphism $\overline{g}_k : P \to Q_k$ with

$$\overline{g} = \overline{\mu}_k \overline{g}_k$$

and we have (6), which implies:

$$g = \overline{g}_k i \ .$$

It means Q_k is P-injective for any P in $\tau[M]$ and for all $k \in I$. So Q_k is $\tau[M]$-injective for all $k \in I$. □

$\tau[M]$-Cohereditary Modules

On this section, we define injective, projective, cohereditary and hereditary of category $\tau[M]$. Moreover we discussed about it is properties and characterization.

Definition 3.1. Let M, Q, P be R-modules.

i. A module Q is called injective if it is M-injective, for all modules M in R-Mod.

ii. A module Q is called cohereditary if every factor module of Q is an injective module in R-Mod.

iii. A module Q is called $\tau[M]$-injective if it is P-injective, for all modules P in $\tau[M]$.

iv. A module Q is called $\tau[M]$-cohereditary (or cohereditary in $\tau[M]$) if $Q \in \tau[M]$ and for any factor module of Q is $\tau[M]$-injective.

Definition 3.2. Let M, Q, P be R-modules.

i. A module P is called projective if it is Q-projective, for all modules Q in R-Mod.

ii. A module P is called hereditary if every submodule of P is a projective module in R-Mod.

iii. A module P is called $\tau[M]$-projective if it is Q-projective, for all modules Q in $\tau[M]$.

iv. A module P is called $\tau[M]$-hereditary if, $Q \in \tau[M]$ and for any submodule of Q is $\tau[M]$-projective.

Lemma 3.3. *Let P, Q be R-modules in* $\tau[M]$. *If Q is a* $\tau[M]$*-injective and every submodule L of P is Q-projective, then Q is* $\tau[M]$*-cohereditary.*

Proof: Let V be a factor module of Q. By Lemma 2.3 (*i*), every factor module of Q is $\tau[M]$-injective. By Definition 3.1 (*iv*), Q is $\tau[M]$-cohereditary. □

This following is dual property of lemma 3.3.

Lemma 3.4. *Let P, Q be R-modules in* $\tau[M]$. *If P is a Q-projective and Q is* $\tau[M]$*-*

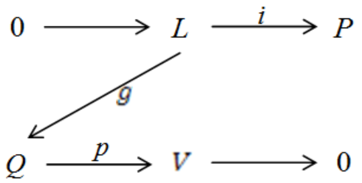
cohereditary, then every submodule L of P is Q-projective.

Proof: Let L be a submodule of P. Since Q is $\tau[M]$-cohereditary, then by definition 3.1 (*iv*) every factor module of Q is P-injective for any P in $\tau[M]$. By lemma 2.3 (*ii*), then every submodule L of P is Q-projective. □

For P and Q R-modules in $\tau[M]$, we obtain properties of $\tau[M]$-cohereditary modules.

Proposition 3.5. *If Q is* $\tau[M]$*-injective, then Q is* $\tau[M]$*-cohereditary.*

Proof: Let L be a submodule of P with P in $\tau[M]$ and let V be a factor module of Q. Suppose g is an R-homomorphism from $L \to Q$. Consider the following diagram:



Since Q is P-injective for any P in $\tau[M]$, then there exists R-homomorphism $\overline{g}: P \to Q$ Q which satisfies:

$$g = \overline{g}i$$

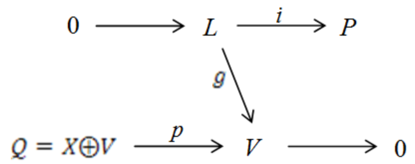
This implies, for any R-homomorphism $pg: L \to V$, there exists an R-homomorphism $p\overline{g}: L \to V$ which satisfy:

$$pg = (p\overline{g})i.$$

It means V is P-injective for any P in $\tau[M]$. By definition 3.1 (*iv*) Q is $\tau[M]$-cohereditary. □

Proposition 3.6. *If Q is* $\tau[M]$*-cohereditary and factor module V is a direct summand of Q, then Q is* $\tau[M]$*-injective.*

Proof: Let L be a submodule of P with P in $\tau[M]$ and let factor module V be a direct summand of Q. Suppose g is an R-homomorphism from $L \to Q$. Consider the following diagram:



Since Q is P-cohereditary for any P in $\tau[M]$ i.e. for any factor module of Q is P injective, then there exists R-homomorphism $\overline{g}: P \to V$ V which satisfy:

$$g = \overline{g}i$$

Also, since V is a direct summand of Q, then there exists an R-homomorphism $\mu: V \to Q$ such that $p\mu = I_v$. This implies, for any R-homomorphism $\mu g: L \to Q$, there exists an R-homomorphism $\mu\overline{g}: P \to Q$ which satisfies:

$$\mu g = (\mu\overline{g})i.$$

It means Q is p-injective for any P in $\tau[M]$. So Q is $\tau[M]$-injective. □

Before we discuss about characterization of $\tau[M]$-cohereditary, we given theorems which is relate with it.

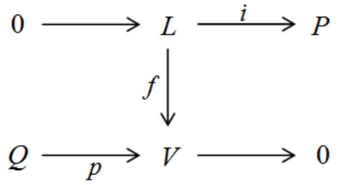
Theorem 3.7. (*Baer's Theorem*) (Passman, 1990) *Every R-module is contained in an injective R-module.*

Theorem 3.8. (Passman, 1990) *An R-module Q is injective if and only if it is a direct summand of every module which contains it.*

Theorem 3.9. *For an injective module Q in* $\tau[M]$*, then the following are equivalent:*

i. *A module Q is a* $\tau[M]$*-cohereditary.*
ii. *For every submodule of a Q-projective in* $\tau[M]$ *is Q-projective.*
iii. *Every factor module of Q is a direct summand of module which contains this factor module.*

Proof: $(i \Rightarrow ii)$ Let L be a submodule of P and let V a factor module of Q with P, Q in $\tau[M]$. Suppose f is an R-homomorphism from $L \to V$. Consider the following diagram:



Since Q is $\tau[M]$-cohereditary i.e. for any V factor module of Q is $\tau[M]$-injective, then there exists R-homomorphism $\overline{f}: P \to V$ which satisfies:

$$f = \overline{f}i \tag{7}$$

Also, since P is a Q-projective for Q in $\tau[M]$, then there exist R-homomorphism $\overline{\overline{f}}: P \to Q$ which satisfies:

$$\overline{f} = p\overline{\overline{f}} \tag{8}$$

By (7) and (8), we have:

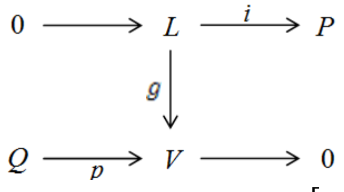
$$f = \overline{f}i = \left(p\overline{\overline{f}}\right)i = p\left(\overline{\overline{f}}i\right).$$

It means, for any R-homomorphism $f: L \to V$, there is exists R-homomorphism $\overline{\overline{f}}i: L \to Q$ which satisfies:

$$f = p\left(\overline{\overline{f}}i\right).$$

This implies L is Q-projective, for every L submodule of P and for Q in $\tau[M]$.

$(i \Leftarrow ii)$ Let L be a submodule of P and let V a factor module of Q. Suppose g is an R-homomorphism from $L \to V$. Consider the following diagram:



Since L is Q-projective for Q in $\tau[M]$, then there exists R-homomorphism $\overline{g}: L \to Q$ which satisfies:

$$g = p\overline{g} \tag{9}$$

Also, since Q is a P-injective for any P in $\tau[M]$, then there exists R-homomorphism $\overline{\overline{g}}: P \to Q$ which satisfies:

$$\overline{g} = \overline{\overline{g}}i \tag{10}$$

By (9) and (10), we have:

$$g = p\overline{g} = p\left(\overline{\overline{g}}i\right) = \left(p\overline{\overline{g}}\right)i.$$

It means, for any R-homomorphism $g: L \to V$, there exists R-homomorphism $p\overline{\overline{g}}: L \to Q$ which satisfy:

$$g = \left(p\overline{\overline{g}}\right)i.$$

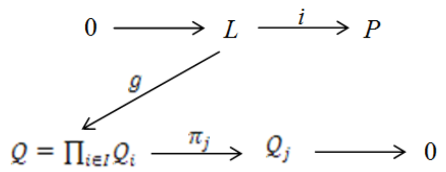
This implies V is P-injective for any P in $\tau[M]$ and for every V factor module of Q. So Q is $\tau[M]$-cohereditary.

$(i \Rightarrow iii)$ Let Q be a $\tau[M]$-cohereditary i.e. for any factor module V of Q is $\tau[M]$-injective. By theorem 3.8, every factor module V is a direct summand of module which contains it.

$(i \Leftarrow iii)$ Let Q be a left R-module in $\tau[M]$ and let V be a factor module of Q. Suppose V is a direct summand of module which contains it. By theorem 3.8, V is $\tau[M]$-injective for any factor module V of Q. By definition 3.1 (*iv*), Q is $\tau[M]$-cohereditary. □

Lemma 3.10. *If* $\{Q_i \mid i \in I\}$ *is a family of* $\tau[M]$*-injective for any* $i \in I$, *then* $Q = \prod_{i \in I} Q_i$ *is* $\tau[M]$*-cohereditary for every index I.*

Proof: Let L be a submodule of P with P in $\tau[M]$ and let $Q = \prod_{i \in I} Q_i$ be a direct product of Q_i. Suppose g is an R-homomorphism from $L \to Q$. Consider the following diagram:



Since Q_j is P-injective for any P in $\tau[M]$ and $j \in I$, then for any R-homomorphism $\pi_j g: L \to Q_j$, there exists an R-homomorphism $\overline{g}_j: P \to Q_j$ which satisfy:

$$\pi_j g = \overline{g}_j i \tag{11}$$

Consider that Q_j is factor module of $Q=\prod_{i\in I}Q_i$ and by (11), so $Q=\prod_{i\in I}Q_i$ is $\tau\left[M\right]$-cohereditary. □

CONCLUSION

In this paper, we showed that injective factor modules and projective submodules are truly dual properties. Also we studied some properties and characterizations of injective and cohereditary modules in category $\tau\left[M\right]$.

Furthermore, all this time the research result that obtained by the author is not able to give meaningful contribution to the algebra structure theory specially modules and rings theory. Nevertheless this written can give new point of view and give way to advanced study for explore some properties and characterizations of projective and hereditary modules in category $\tau\left[M\right]$.

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