On Total Vertex Irregularity Strength of Cocktail Party Graph

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ABSTRACT

A vertex irregular total k-labeling of a graph G is a function λ from both the vertex and the edge sets to $\{1, 2, 3, \cdots, k\}$ such that for every pair of distinct vertices u and x, $\lambda(u) + \sum_{uv \in E} \lambda(uv) \neq \lambda(x) + \sum_{xy \in E} \lambda(xy)$. The integer k is called the total vertex irregularity strength, denoted by $tvs(G)$, is the minimum value of the largest label over all such irregular assignments. In this paper, we prove that the total vertex irregularity strength of the Cocktail Party graph $H_{2,n}$, that is $tvs(H_{2,n}) = 3$ for $n \geq 3$.

Keywords : Total vertex irregularity strength, cocktail party graph

INTRODUCTION

Throughout this paper all graphs are finite, simple, and undirected. A vertex irregular total *k*-labeling on a graph *G* is a function λ from both the vertex and the edge sets to $\{1, 2, 3, \cdots, k\}$ such that the weights calculated at vertices are distinct. The weight of a vertex *v* in *G* is defined as the sum of the label of *v* and the labels of all the edges incident with *v*, that is, $wt(uv) = \lambda(u) + \sum_{uv \in E} \lambda(uv)$.

The notion of the vertex irregular total *k*-labeling was introduced by Bača, *et al*. (2007). The total vertex irregularity strength of *G*, denoted by $tvs(G)$, is the minimum value of the largest label over all such irregular assignments.

Many total vertex irregularity strength of *G* had already been identified for some classes of graph. Bača, *et al*. (2007) proved that if a tree *T* with *n* pendant vertices and no vertices of degree 2, then $\left\lceil \frac{n+1}{2} \right\rceil \leq tvs(T) \leq n$. Additionally, they gave a lower and an upper bound on total vertex irregular strength for any graph *G* with *v* vertices and *e* edges, minimum degree δ and maximum degree Δ, $\left\lceil \frac{|V| + \delta}{\Delta + 1} \right\rceil \leq tvs(G) \leq |V| + \Delta - 2\delta - 1$. In the same paper, they proved that $tvs(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$, $tvs(K_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil$, and $tvs(K_n) = 2$. Furthermore, the total vertex irregularity strength of complete bipartite graphs $K_{m,n}$ for some *m* and *n* had been discovered by Wijaya *et al*. (2005), namely, $tvs(K_{2,n}) = \left\lceil \frac{n+2}{3} \right\rceil$ for $n > 3$, $tvs(K_{n,n}) = 3$ for $n \geq 3$, $tvs(K_{n,n+1}) = 3$ for $n \geq 3$, $tvs(K_{n,n+2}) = 3$ for $n \geq 4$, and $tvs(K_{n,an}) = \left\lceil \frac{n(a+1)}{n+1} \right\rceil$ for all *n* and $a > 1$. Besides, they gave the lower bound on $tvs(K_{m,n})$ for $m < n$, that is $tvs(K_{m,n}) \geq \max\left\{ \left\lceil \frac{m+n}{m+1} \right\rceil, \left\lceil \frac{2m+n-1}{n} \right\rceil \right\}$. Wijaya and Slamin (2008) found the values of total vertex irregularity strength of wheels W_n, fans F_n, suns S_n and friendship graphs f_n by showing that $tvs(W_n) = \left\lceil \frac{n+3}{4} \right\rceil$, $tvs(F_n) = \left\lceil \frac{n+2}{4} \right\rceil$, $tvs(S_n) = \left\lceil \frac{n+1}{2} \right\rceil$, $tvs(f_n) = \left\lceil \frac{2n+2}{3} \right\rceil$. Ahmad *et al*. (in press) had determined total vertex irregularity strength of Halin graph. Whereas the total vertex irregularity strength of several types of trees determined by Nurdin *et al*. (to appears) and the total vertex irregularity strength of disjoint union of *t* copies of a path had been determined by Nurdin *et al*. (2009). Ahmad & Bača (Gallian 2009) proved that $tvs(J_{n,2}) = \left\lceil \frac{n+1}{2} \right\rceil$ for $n \ge 4$ and conjectured that for $n \ge 3$ and $m \ge 3$, $tvs(J_{m,n}) \ge \max\left\{ \left\lceil \frac{n(m-1)+2}{3} \right\rceil, \left\lceil \frac{nm+2}{4} \right\rceil \right\}$. They also proved that for the circulant graph, $tvs(C_n(1,2)) = \left\lceil \frac{n+4}{5} \right\rceil$, and conjectured that for the circulant graph $C_n(a_1, a_2, \cdots, a_m)$ with degree at least 5, $1 \le a_i \le \left\lfloor \frac{n}{2} \right\rfloor$, $tvs(C_n(a_1, a_2, \cdots, a_m)) = \left\lceil \frac{n+r}{1+r} \right\rceil$.

In this paper, we determine the total vertex irregularity strength of the cocktail party graph $H_{2,n}$ for $n \ge 3$, as described in the following section.

RESULTS AND DISCUSSION

The cocktail party graph $H_{2,n}$, $n \ge 3$ is a graph with a vertex set $V = \{v_1, v_2, \cdots, v_{2n}\}$ partitioned into n independent sets $V = \{I_1, I_2, \cdots, I_n\}$ each of size 2 such that $v_i v_j \in E$ for all $i, j \in \{1, 2, \cdots, 2n\}$ where $i \in I_p$, $j \in I_q$, $p \ne q$. It is the complement of a disjoint union of n copies of K_2. We present the total vertex irregularity strength of the cocktail party graph $H_{2,n}$ for $n \ge 3$, as follows.

Theorem 1. $tvs(H_{2,n}) = 3$ for $n \ge 3$.

Proof. Let $H_{2,n}$ be the cocktail party graph with $n \ge 3$. Then $H_{2,n}$ has 2n vertices of degree $2n-2$. The smallest weight of vertices of $H_{2,n}$ must be $2n-1$ and the largest weight of vertices of $H_{2,n}$ is at least $4n-2$. Since $4n-2$ is obtained from the sum of $2n-1$ numbers, then at least all edges and one vertex must have label $\frac{4n-2}{2n-1} = 2$. This is impossible because there must be at least one edge with label 1 to obtain lower weight than $4n-2$. Thus $tvs(H_{2,n}) \ge 3$.

To show that $tvs(H_{2,n}) \le 3$, we label the vertices and edges of $H_{2,n}$ as a vertex irregular total 3-labeling. Suppose the cocktail party graph has the set of vertices,

$V(H_{2,n}) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ for $i = 1, 2, \cdots, n$ and the set of edges,

$E(H_{2,n}) = \{u_i u_j\}_{i \ne j} \cup \{u_i v_j\}_{i \ne j} \cup \{v_i v_j\}_{i \ne j}$

The labels of the vertices of $H_{2,n}$ are described in the following formulas:

$\lambda(u_i) = 1$, for $i = 1, 2, \cdots, n.$.

$$\lambda(v_i) = \begin{cases} 1, & \text{for } \; i = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor \\ 2, & \text{for } \; i = \left\lceil \frac{n}{2} \right\rceil, \text{ if } n \text{ is odd} \\ 3, & \text{for } \; i = \left\lceil \frac{n}{2} \right\rceil + 1, \left\lceil \frac{n}{2} \right\rceil + 2, \cdots, n. \end{cases}$$

The labels of edges of $H_{2,n}$ are as follows.

For $i \ne j$ = j, where $i, j = 1, 2, \cdots, n$.

$\lambda(u_i u_j) = 1$,

$\lambda(v_i v_j) = 2$,

$$\lambda(u_i v_j) = \begin{cases} 1, & \text{if } i < j \text{ and } i + j \le n+1, \\ 1, & \text{if } i > j \text{ and } i + j \le n, \\ 2, & \text{if else.} \end{cases}$$

The weights of vertices u_i and v_i are:

$wt(u_i) = 2n - 2 + i$ for $i = 1, 2, \cdots, n$;

$wt(v_i) = 3n - 2 + i$ for $i = 1, 2, \cdots, n$.

It is easy to see that the weights calculated at vertices are distinct. So, the labelling is vertex irregular total. Therefore $tvs(H_{2,n}) = 3$ for $n \ge 3$. □

As an illustration, we show the vertex irregular total 3-labeling of $H_{2,6}$ (in Table 1) and $H_{2,7}$ (in Table 2). We note that label 0 means that there is no edge connecting two vertices.

Table 1. The vertex irregular total 3-labeling of $H_{2,6}$.

$H_{2,6}$		u_1	u_2	u_3	u_4	u_5	u_6	v_1	v_2	v_3	v_4	v_5	v_6	*wt*
		1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**3**	**3**	**3**	
u_1	**1**	0	1	1	1	1	1	0	1	1	1	1	1	11
u_2	**1**	1	0	1	1	1	1	1	0	1	1	1	2	12
u_3	**1**	1	1	0	1	1	1	1	1	0	1	2	2	13
u_4	**1**	1	1	1	0	1	1	1	1	2	0	2	2	14
u_5	**1**	1	1	1	1	0	1	1	2	2	2	0	2	15
u_6	**1**	1	1	1	1	1	0	2	2	2	2	2	0	16
v_1	**1**	0	1	1	1	1	2	0	2	2	2	2	2	17
v_2	**1**	1	0	1	1	2	2	2	0	2	2	2	2	18
v_3	**1**	1	1	0	2	2	2	2	2	0	2	2	2	19
v_4	**3**	1	1	1	0	2	2	2	2	2	0	2	2	20
v_5	**3**	1	1	2	2	0	2	2	2	2	2	0	2	21
v_6	**3**	1	2	2	2	2	0	2	2	2	2	2	0	22
wt		11	12	13	14	15	16	17	18	19	20	21	22	

Table 2. The vertex irregular total 3-labeling of $H_{2,7}$.

$H_{2,7}$		u_1	u_2	u_3	u_4	u_5	u_6	u_7	v_1	v_2	v_3	v_4	v_5	v_6	v_7	*wt*
		1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**2**	**3**	**3**	**3**	
u_1	**1**	0	1	1	1	1	1	1	0	1	1	1	1	1	1	13
u_2	**1**	1	0	1	1	1	1	1	1	0	1	1	1	1	2	14
u_3	**1**	1	1	0	1	1	1	1	1	1	0	1	1	2	2	15
u_4	**1**	1	1	1	0	1	1	1	1	1	1	0	2	2	2	16
u_5	**1**	1	1	1	1	0	1	1	1	1	2	2	0	2	2	17
u_6	**1**	1	1	1	1	1	0	1	1	2	2	2	2	0	2	18
u_7	**1**	1	1	1	1	1	1	0	2	2	2	2	2	2	0	19
v_1	**1**	0	1	1	1	1	1	2	0	2	2	2	2	2	2	20
v_2	**1**	1	0	1	1	1	2	2	2	0	2	2	2	2	2	21
v_3	**1**	1	1	0	1	2	2	2	2	2	0	2	2	2	2	22
v_4	**2**	1	1	1	0	2	2	2	2	2	2	0	2	2	2	23
v_5	**3**	1	1	1	2	0	2	2	2	2	2	2	0	2	2	24
v_6	**3**	1	1	2	2	2	0	2	2	2	2	2	2	0	2	25
v_7	**3**	1	2	2	2	2	2	0	2	2	2	2	2	2	0	26
wt		13	14	15	16	17	18	19	20	21	22	23	24	25	26	

CONCLUSION

We conclude this paper with a result that the total vertex irregularity strength of the Cocktail Party graph $H_{2,n}$, that is $tvs(H_{2,n}) = 3$ for $n \geq 3$. We also give an open problem, that is, determining the total vertex irregularity strength of generalization of cocktail party graph $H_{m,n}$ for $m > 2$ and $n \geq 3$

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