On The Graphs and Their Complements with Prescribed Circumference

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ABSTRACT

Let $\mathsf{G}_t(n)$ be the class of connected graphs on n vertices having the longest cycle of length t and let $G \in \mathsf{G}_t(n)$. Woodall (1976) determined the maximum number of edges of G. An alternative proof and characterization of the extremal (edge-maximal) graphs given by Caccetta & Vijayan (1991). The edge-maximal graphs have the property that their complements are either disconnected or have a cycle going through each vertex (i.e. they are hamiltonian). This motivates us to investigate connected graphs with prescribed circumference (length of the longest cycle) having connected complements with cycles . More specifically, we focus our investigations on the class G (n, c, $\bar{c}$) denoting the class of connected graphs on n vertices having circumference c and whose connected complements have circumference $\bar{c}$. The problem of interest is that of determining the bounds of the number of edges of a graph $G \in \mathsf{G}$(n, c, $\bar{c}$) and characterize the extremal graphs of G(n, c, $\bar{c}$).
We discuss the class G (n, c, $\bar{c}$) and present some results for small c. In particular for c = 4 and $\bar{c}$ = n - 2, we provide a complete solution.

Keywords : Extremal graph, circumference

INTRODUCTION

The properties of the graphs usually involve certain graph parameters. A great deal of graph theory is concerned with establishing the best bounds for graph parameters and characterizing the graphs for which the bounds are achieved. This important area of graph theory, called extremal graph theory, forms the main focus of this paper. The property we consider is expressed in terms of the length of the largest cycle in the graph and the length of the largest cycle in the graphs complement. In particular, we focus our attention on the problem of determining the bounds of the number of edges of graph G if given c(G) and c($\overline{G}$).

We use standard set theoretic notation and terminology. As there is considerable variation in the graph theoretic notation and terminology used in the literature, we present, in this section, the basic notation and terminology that we use in this paper. For the most part, our notation and terminology follows that of Bondy & Murty (1976). We denote the vertex set of a graph G by $V(G)$ and the edge set of G by $E(G)$; the cardinalities of these sets are denoted by ν (G) and ε (G), respectively. We use the standart notation denoting the complete graph on n vertices by K_n and the complete bipartite graph with bipartitioning sets of order m and n by $K_{m,n}$. The path and cycle on n vertices are denoted by P_n and C_n, respectively. The join between two graphs G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining every vertex of G to every vertex of H.

Let G_t(n) be the class of connected graphs on n vertices having the longest cycle of length t and let $G \in G_t(n)$. Woodall (1976) determined the maximum number of edges of G. An alternative proof and characterization of the extremal (edge-maximal) graphs given by Caccetta & Vijayan (1991).

The following three lemmas formed an important component of the method of proof given by Caccetta & Vijayan (1991).

Lemma 1.1
Let $G \in G_t(n)$ and let $x \in H$ be joined to the vertices i_1, i_2, ..., i_k of C. Then, for $1 \le \alpha \ne \beta \le k$, we have :
(a) $|i_\alpha - i_\beta| \ge 2$;
(b) $(i_\alpha - 1,\ i_\beta - 1), (i_\alpha + 1,\ i_\beta + 1) \notin E(G)$.

This lemma tells us that any vertex of $G - V(C)$ cannot be joined to two consecutives vertices of a cycle C in G.

Lemma 1.2
Let $G \in G_t(n)$ and let $P = i, x_1, x_2, \ldots, x_{d-1}, j$ be an (i, j)-path, $i \ne j$, of length d whose internal

vertices are not in C. Then (a) $d \leq |i - j| \leq t - d$, so $t \geq 2d$,
(b) for positive integers a, b with $a + b \leq d$: $(i + a, j + b), (i - a, j - b) \notin E(G)$.

Instead of a single vertex of $G - V(C)$, this lemma considers a path of length d whose internal vertices are not in C. So, this result generalizes Lemma 1.1.

Lemma 1.3

Let $G \in \mathsf{G}_t(n)$ and let $P = i, x_1, x_2, \ldots, x_{d-1}, j$ be an (i, j)-path, $i \neq j$, of length $d \geq 2$ whose internal vertices are not in C. Suppose x_1 is joined to k vertices of C.
Let $A = G[V(C)]$. Then
$\varepsilon(A) \leq \frac{1}{2} t(t-1) - \frac{1}{2}(k + d - 2)(k + d - 3)$,
with equality holding only if $d = 2$ and $t = 2k$.

We often make use of the above three lemmas since our focus in this paper is to determine the maximum number of edges of a graph G having certain properties. Our properties specify that G and its complement have a longest cycle of specified length and both must be connected.

The following lemma was given by Xu (1987).

Lemma 1.4

Let G be a graph of order $n \geq 6$, and both G and $\overline{G}$ have cycles, then

$n + 2 \leq c(G) + c(\overline{G}) \leq 2n$

and

$3(n - 1) \leq c(G) \cdot c(\overline{G}) \leq n^2$

The following lemma is due to Kusmayadi and Caccetta (2001).

Lemma 1.5

Let $G \in \mathsf{G}_{2k}(n)$, $k \geq 2$, be a k-connected graph. Then $\overline{G}$ is not connected.

We now consider the class $\mathsf{G}(n, 4, n-2)$ of connected graphs having a cycle of length 4 and a connected complement with a cycle of length $n - 2$. Let $G \in \mathsf{G}(n, 4, n-2)$. Kusmayadi and Caccetta (2001) found the bounds of $\varepsilon(G)$ as stated in the following theorem.

Theorem 1.6

Let $G \in \mathsf{G}(n, 4, n-2)$, $n \geq 9$. Then $\varepsilon(G) = 2n - 4$ or $2n - 5$.
Moreover, these bounds are sharp.

METHODS

The methods used to do this research are a literature study. In particular, it is used the knowledge owned by the authors from the previous research experiences. The strategy used in this research is divided into some steps as the following. Let $G \in \mathsf{G}(n, 4, n-2)$.

The first step is investigating the diameter of G, $d(G)$, by considering the properties of a graph G in the class of $\mathsf{G}(n, 4, n-2)$. The second step is to investigate the minimum degree of G, $\delta(G)$, by considering the properties of a graph G in the class of
$\mathsf{G}(n, 4, n-2)$. The next step is determining the number of edges of $\overline{A} = \overline{G}[C]$, where C is a cycle of length $n - 2$. The final step is characterising the graph $G \in \mathsf{G}(n, 4, n-2)$.

RESULTS AND DISCUSSION

The main goal of this section is giving the characterization of the extremal graphs of $\mathsf{G}(n, 4, n-2)$. Let $G \in \mathsf{G}(n, 4, n-2)$. The following few results deal with the diameter $d(G)$ of $G \in \mathsf{G}(n, 4, n-2)$.

The diameter d(G) of $G \in \mathsf{G}(n,4,n-2)$

Lemma 3.1

Let $G \in \mathsf{G}(n, 4, n-2)$, $n \geq 11$. Then $d(G) \geq 3$.

Proof :

Let $G \in \mathsf{G}(n, 4, n-2)$. Then , by Lemma 1.5, G has a cut vertex, v say. Suppose that $d(G) \leq 2$. Then every vertex of G is adjacent to v and hence $d_G(v) = n - 1$. But then $\overline{G}$ cannot be connected. Hence $d(G) \geq 3$.

Lemma 3.2

Let $G \in \mathsf{G}(n, 4, n-2)$, $n \geq 11$. Then $d(G) \leq 3$.

Proof :

Suppose that $d(G) \geq 4$ and let G be the smallest graph on n vertices satisfying the hypothesis in the lemma. We will prove that $\delta(G) \geq 2$.

Suppose $\delta(G) = 1$ and let $d_G(x) = 1$. Consider $G - x$. Clearly, $G - x$ is connected, $c(G - x) = c(G) = 4$, $\varepsilon(G - x) = \varepsilon(G) - 1$ and $d(G - x) \leq d(G)$. By Lemma 1.4, in $\overline{G - x}$, we have :

$c(G - x) + c(\overline{G - x}) \geq (n - 1) + 2 = n + 1$.

Therefore

$c(\overline{G - x}) \geq (n + 1) - c(G - x) = n - 3$ (since $c(G - x) = 4$).

By the choice of G, we know that

$c(\overline{G - x}) \neq (n - 1) - 2$.

Since $c(\overline{G - x}) \leq c(\overline{G}) = n - 2$ and $c(\overline{G - x}) \neq n - 3$, then the only possibility is that $c(\overline{G - x}) = n - 2$.

But then we have $c(\overline{G}) > n - 2$ (since $d_G(x) = 1$), a contradiction. So we must have $\delta(G) \geq 2$.

From theorem 1.6, we know that $\varepsilon(G) = 2n - 4$ or $2n - 5$. So, the average degree $\overline{d}$ of G is $\overline{d} = 2\varepsilon(G)/n \leq (4n - 8)/n < 4$.

This implies that $\delta(G) \leq 3$ and hence $\delta(G) = 2$ or 3.

Suppose $d_G(x) = \delta$. Consider $G - x$. Obviously, $c(G - x) \leq c(G)$. We will show that

$c(G - x) = c(G)$.

In $G - x$, we have

$\varepsilon(G - x) = \varepsilon(G) - \delta \geq 2n - 5 - \delta = 3(n - 2)/2 + n/2 - 2 - \delta \geq 3(n - 2)/2 + (n - 10)/2 > 3(n - 2)/2$.

Therefore, $c(G - x) \geq 4$ and so $c(G - x) = c(G) = 4$, as required.

We claim that $c(\overline{G - x}) \neq c(\overline{G}) - 1$. Suppose $c(\overline{G - x}) = c(\overline{G}) - 1$. Then $c(\overline{G - x}) = (n - 2) - 1 = n - 3$. Since $d_G(x) = \delta \leq 3$, then $d_{\overline{G}}(x) \geq n - 4$. Consider $\overline{G}$.

The situation is as depicted in the following Figure 1.

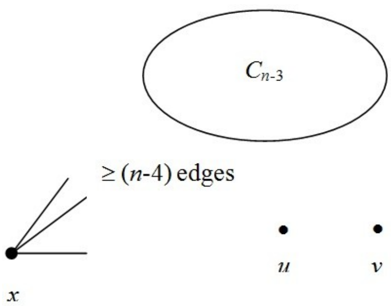

Figure 1 . $\overline{G}$

We consider four cases according to the values of $e(x, C_{n-3})$, the number of edges coming from the cycle C_{n-3} to vertex x.

Case 1 : $e(x, C_{n-3}) = n - 3$.

Since the maximum degree of x in $\overline{G}$ is at most $n - 3$, then x cannot be joined to vertices u and v of $\overline{G}$. In addition, vertices u and v are joined to at most one vertex of C_{n-3}, as otherwise $c(\overline{G}) > n - 2$. Therefore

$\varepsilon(\overline{G}) \leq ½ (n - 3)(n - 4) + (n - 3) + 3 \leq ½ (n^2 - 5n + 12)$,

with equality achieved when $\overline{G}$ is as shown in Figure 2.

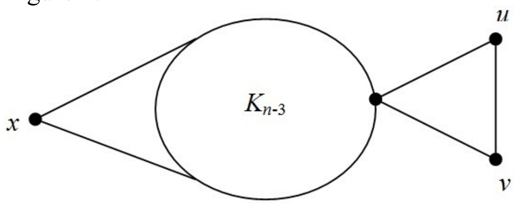

Figure 2 . $\overline{G}$

But then G is disconnected. Therefore

$\varepsilon(\overline{G}) \leq ½ (n^2 - 5n + 10)$.

Now, we claim that there exists vertices c_i and c_j of $V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$.

Suppose not. Then $\overline{G}[V(C_{n-3}) \cup \{x\}] \cong K_{n-2}$ and hence $d(G) < 4$, a contradiction. Therefore, there exists c_i and c_j in $V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$.

Suppose $c_i, c_j \in N_{C_{n-3}}(u)$. Then, in $\overline{G}$, we have a cycle C : $x, c_{j+1}, c_{j+2}, \ldots, c_i, u, c_j, c_{j-1}, \ldots, c_{i+1}, x$ of length $n - 1$, a contradiction. This implies that u and v are each joined to at most one of c_i and c_j.

Now, suppose $c_i \in N_{C_{n-3}}(u) \cap N_{C_{n-3}}(v)$. It is easy to check that, in G, we have $d(G) = 2$, a contradiction. So, without no loss of generality, we can assume that $vc_i \notin E(\overline{G})$ and $uc_j \notin E(\overline{G})$. But then, in G, we have a cycle C : u, c_k, v, c_i, c_j, u of length 5, a contradiction.

Case 2 : $e(x, C_{n-3}) = n - 4$.

Then $d_{\overline{G}}(x) = n - 4$ or $n - 3$. We consider these two possibilities separately.

Suppose that $d_{\overline{G}}(x) = n - 4$. Then x cannot be joined to vertices u and v of $\overline{G}$. In addition, u and v can only be joined to at most one vertex of C_{n-3}. The reason for this is as follows : Suppose, without no loss of generality, uc_i and $uc_j \in E(\overline{G})$. Then, in $\overline{G}$, we have a cycle C: $x, c_{j+1}, c_{j+2}, \ldots, c_i, u, c_j, c_{j-1}, \ldots, c_{i+1}, x$ of length $n - 1$,

a contradiction. Hence,

$\varepsilon\ (\overline{G}) \leq$ ½ $(n-3)(n-4) + (n-4) + 3 \leq$ ½ $(n^2 - 5n + 10)$,
with equality achieved when $\overline{G}$ is as shown in Figure 3.

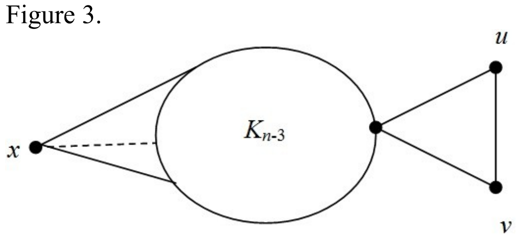

Figure 3. $\overline{G}$

But then G is disconnected or $d(G) = 2$, a contradiction.
Therefore,
$\varepsilon\ (\overline{G}) \leq$ ½ $(n^2 - 5n + 8)$.
Again, we claim that there exists c_i and c_j of $V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$. Suppose not. Let $e = xc_r$, where $c_r \in V(C_{n-3})$. Then $\overline{G}[V(C_{n-3}) \cup \{x\}] \cong K_{n-2}\backslash e$ and hence $d(G) < 4$, a contradiction. Therefore, there exists c_i, $c_j \in V(C_{n-3})$ such that $c_ic_j \notin E(\overline{G})$.

Suppose c_i, $c_j \in N_{C_{n-3}}(u)$. Then, in $\overline{G}$, we have a cycle C : x, c_{j+1}, c_{j+2}, …, c_i, u, c_j, c_{j-1}, …, c_{i+1}, x of length $n-1$, a contradiction. This implies that u and v are each joined to at most one of c_i and c_j.
Now, suppose $c_i \in N_{C_{n-3}}(u) \cap N_{C_{n-3}}(v)$. It is easy to check that, in G, we have $d(G) = 2$ or there is a cycle C : x, c_r, u, c_j, v, x (note that c_r could be the same as c_i) of length 5, a contradiction. So, with no loss of generality, we can assume that $vc_i \notin E(\overline{G})$ and $uc_j \notin E(\overline{G})$. But then, in G, we have a cycle C : u, c_k, v, c_i, c_j, u of length 5, a contradiction.

Suppose now that $d_{\overline{G}}(x) = n-3$. Then, without loss of generality, we may assume that $ux \in E(\overline{G})$. Clearly, u cannot be joined to any vertices of C_{n-3} and vertex v can only be joined to at most one vertex of C_{n-3}, as otherwise, suppose vc_i and $vc_j \in E(\overline{G})$. Then, in $\overline{G}$, we have a cycle C : x, c_{j+1}, c_{j+2}, …, c_i, u, c_j, c_{j-1}, …, c_{i+1}, x of length $n-1$, a contradiction.
But then, in G, we can find a cycle C : u, c_1,v, x, c_2, u (c_1, $c_2 \in V(C_{n-3})$) of length 5, a contradiction.

Case 3 : $e(x, C_{n-3}) = n-5$. u
If $d_{\overline{G}}(x) = n-3$, then x must also be joined to u and v. Clearly, u and v cannot be joined to any vertex of C_{n-3}, as otherwise $c(\overline{G}) > n-2$. But then, in G, we have a cycle C : u, c_1,v, c_2, x, c_3, u ($c_i \in V(C_{n-3})$, $i = 1, 2$) of length 6, a contradiction.
If $d_{\overline{G}}(x) = n-4$, then, without loss of generality, we may assume that $ux \in E(\overline{G})$. Clearly, u cannot be joined to any vertices of C_{n-3} and vertex v can only be joined to at most one vertex of C_{n-3}, as otherwise $c(\overline{G}) > n-2$. Again, in G, we have a cycle C : u, c_1,v, c_2, x, c_3, u ($c_i \in V(C_{n-3})$, $i = 1, 2, 3$) of length 6, a contradiction.
Case 4 : $e(x, C_{n-3}) = n-6$.
Clearly $d_{\overline{G}}(x) = n-4$ and hence x must also be joined to u and v. In addition, u and v cannot be joined to any vertex of C_{n-3}, as otherwise $c(\overline{G}) > n-2$. But then, in G, we have a cycle C : u, c_1,v, c_2, x, c_3, u ($c_i \in V(C_{n-3})$) of length 6, a contradiction.
Therefore, $c(\overline{G-x}) = c(\overline{G})$ or $c(\overline{G-x}) \leq c(\overline{G}) - 2$.
If $c(\overline{G-x}) = n-2$ and since $d_{\overline{G}}(x) \geq n-4$, then $c(\overline{G}) > n-2$, a contradiction.
Now, if $c(\overline{G-x}) \leq (n-2) - 2 = (n-1) - 3$ and since $d_{\overline{G}}(x) = n-3$ or $n-4$, then clearly that $c(\overline{G}) < n-2$, a contradiction. This completes the proof of the lemma. €
Lemmas 3.1 and 3.2 together give :
Theorem 3.3
Let $G \in$ G$(n, 4, n-2)$, $n \geq 11$. Then d(G) = 3. €

The minimum degree δ (*G*) of *G*∈ G(*n*,4,*n*–2)
The following result deals with the minimum degree $\delta\ (G)$ of a graph $G \in$ G$(n, 4, n-2)$.
Lemma 3.4
Let $G \in$ G$(n, 4, n-2)$, $n \geq 11$. Then $\delta\ (G) = 1$.
Proof :

Let G be the smallest graph on $n \geq 11$ vertices satisfying the hypothesis in the lemma. By Theorem 1.6, the average degree $\overline{d}$ of G is

$\overline{d} = 2\varepsilon(G)/n \leq (4n-8)/n < 4$,

and so $1 \leq \delta(G) \leq 3$.

Suppose $\delta(G) \geq 2$ and $d_G(x) = \delta$. Consider $G - x$. Clearly, $c(G-x) \leq c(G) = 4$.

By Lemma 1.4 we have $c(G-x) + c(\overline{G-x}) \geq (n-1) + 2 = n + 1$,

and hence

$c(\overline{G-x}) \geq (n+1) - c(G-x) = (n+1) - 4$ (Since $c(G-x) \leq 4$) $= n - 3$.

Obviously, $c(\overline{G-x}) \leq c(\overline{G}) = n - 2$.

By the choice of G, we know that $c(\overline{G-x}) \neq (n-1) - 2$.

So the only possibility is $c(\overline{G-x}) = n - 2$. Since $d_G(x) = 2$ or 3 then we have $c(G) > n - 2$, a contradiction. This completes the proof of the lemma.

The number of edges $\boldsymbol{\varepsilon}$ ($\overline{A}$) where $\overline{A} = \overline{G}$ *[C]*

Let $G \in \mathcal{G}(n, 4, n-2)$ and let C be a cycle of length $n - 2 = c(\overline{G})$ in the connected complement $\overline{G}$. Suppose $\overline{A} = \overline{G}[C]$. The following lemma gives the lower bound of the number of edges of $\overline{A}$.

Lemma 3.5

Let $G \in \mathcal{G}(n, 4, n-2)$ and let C be a cycle of length $n - 2 = c(\overline{G})$ in the connected complement $\overline{G}$. Suppose $\overline{A} = \overline{G}[V(C)]$. Then $\varepsilon(\overline{A}) \geq \binom{n-2}{2} - 1$.

Proof :

Let $G \in \mathcal{G}(n, 4, n-2)$ and let $C = \{x_1, x_2, \ldots, x_{n-2}\}$ be a cycle of length $n - 2$ in $\overline{G}$. Let $\overline{H} = \overline{G} - V(C) = \{u_1, u_2\}$ and $\overline{A} = \overline{G}[V(C)]$. Consider $\overline{G}$. We, first, show that $d_{\overline{G}}(u) \leq 2$ for any $u \in V(\overline{H})$. Suppose $d_{\overline{G}}(u) \geq 3$ for some $u \in V(\overline{H})$. Then, at least one of the vertices of $\overline{H}$ must be joined to at least two vertices of C. Suppose $u_1 x_i$ and $u_1 x_j \in E(\overline{G})$.

By Lemma 1.1, we get $|i - j| \geq 2$ and $x_{i+1} x_{j+1} \notin E(\overline{G})$.

Now, suppose $N_{\overline{G}}(u_1) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$.

By Lemmas 1.1 and 1.2, we get

$$x_{i_1+1} x_{i_m+1} \notin E(\overline{G}), 1 \quad l \quad m \quad k.$$

This implies that, in G, we can find a path P_k and vertex u_1 is joined to every vertex of this path P_k. If $k \geq 4$, we can get a cycle of length at least 5 in G, a contradiction.

Therefore, $|N_{\overline{G}}(u_1) \cap C| \leq 3$.

If $|N_{\overline{G}}(u_1) \cap C| = 2$ and $d_{\overline{G}}(u_2) \leq 2$, then by Lemmas 1.1 and 1.2, we can find, in G, an edge e and vertex x_k such that e is incident to u_1 and u_2 and both u_1 and u_2 are joined to vertex x_k. Consequently, there exists a cycle $u_1 x_{i+1} x_{j+1} u_2 x_k u_1$ of length 5 in G, a contradiction.

Now, suppose $|N_{\overline{G}}(u_1) \cap C| = 3$ and $d_{\overline{G}}(u_2) \leq 2$. Again, by Lemmas 1.1 and 1.2, we can find a path P_3 in G such that vertex u_1 is joined to every vertex of P_3 and vertex u_2 is joined to at least two vertices of P_3. Hence, we get a cycle of length 5 in G, a contradiction.

So, the only possibility is $d_{\overline{G}}(u_1) = d_{\overline{G}}(u_2) = 3$. Without any loss of generality, the situation can be depicted as shown in Figure 4.

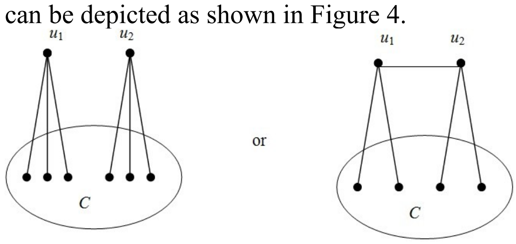

Figure 4. $\overline{G}$

If $u_1u_2 \notin E(\overline{H})$, then clearly $|N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2)| \leq 3$. Hence, we can find a path P_3 in G such that vertices u_1 and u_2 are joined to every vertex of this path P_3. Therefore, we get a cycle of length at least 5 in G, a contradiction.

Now, if $u_1u_2 \in E(\overline{H})$, then $|N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2)| \leq 2$. If $|N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2)| = 0$, then, in G, we can find two K_2's such that vertices u_1 and u_2 are joined to every vertex of these K_2's. Hence, we get a cycle of length at least 5 in G, a contradiction.

If $|N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2)| \geq 1$, then we can find K_2 and a vertex x_k in G such that u_1 and u_2 are joined to x_k and every vertex of K_2. This implies that G has a cycle of length 5 : $u_1\, x_{i+1}$

$x_{j+1}\, u_2\, x_k\, u_1$, a contradiction. Therefore, we have $d^-_{\overline{G}}(u) \leq 2$ for any vertex $u \in V(\overline{H})$.

If vertices u_1 and u_2 of $\overline{H}$ are joined to the same vertex of C, then by Lemma 3.4, we have $\Delta(\overline{G}) = n-2$, and hence

$$\varepsilon(\overline{A}) = \binom{n-2}{2} - 1. \qquad (1)$$

If vertices u_1 and u_2 of $\overline{H}$ are joined to the different vertex of C, then again, by Lemma 3.4, we have $\Delta(\overline{G}) = n-2$, and then

$$\varepsilon(\overline{A}) = \binom{n-2}{2}. \qquad (2)$$

From (1) and (2) we get $\varepsilon(\overline{A}) \geq \binom{n-2}{2} - 1$, as required.

Remark 3.1

Let $G \in \mathsf{G}(n, 4, n-2)$ and let $\overline{G}$ be the connected complement of G. Then $d(\overline{G}) = 3$. This follows from Theorem 3.3 and Lemma 3.5.

The characterisation of $G \in \mathsf{G}(n, 4, n-2)$

We are now ready to characterize the extremal graphs of $\mathsf{G}(n, 4, n-2)$ as stated in the following theorem.

Theorem 3.6

Let $G \in \mathsf{G}(n, 4, n-2)$. Then $G \cong G_i$, $i = 1, 2, 3, 4$,

where $\varepsilon(G_i) = \begin{cases} 2n-5, & i = 1, 2, 3 \\ 2n-4, & i = 4 \end{cases}$

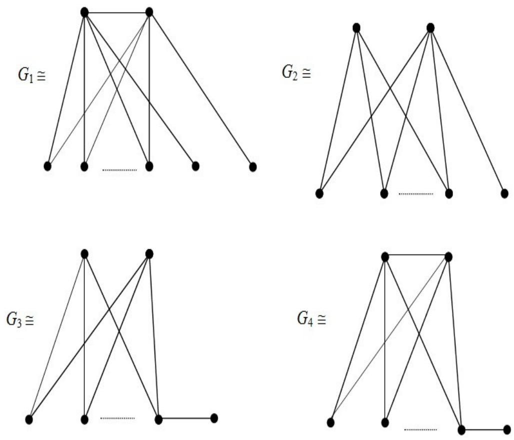

Figure 5. The extremal graphs of $G \in \mathsf{G}(n,4,n-2)$

Proof :

Let $G \in \mathsf{G}(n, 4, n-2)$ and let $\overline{G}$ be the connected complement of G having a cycle C of length $n-2 = c(\overline{G})$ in $\overline{G}$. Let $C = \{x_1, x_2, \ldots, x_{n-2}\}$, $\overline{A} = \overline{G}[V(C)]$ and $\overline{H} = \overline{G} - V(C) = \{u_1, u_2\}$.

By Lemma 3.5, $\varepsilon(\overline{A}) \geq \binom{n-2}{2} - 1$.

So, we have two cases to consider concerning the number of edges in $\overline{A}$.

Case 1 : $\varepsilon(\overline{A}) = \binom{n-2}{2}$.

Then $\overline{A} \cong K_{n-2}$. Therefore, any vertex of $\overline{H}$ can be joined to at most one vertex of $\overline{A}$, as otherwise $c(\overline{G}) > n-2$.

If $u_1u_2 \in E(\overline{G})$, then without loss of generality, we can take $u_1x_i \in E(\overline{G})$ and $u_2x_j \notin E(\overline{G})$, $1 \leq j \leq n-2$. So, we get $G \cong G_2$. If $u_1u_2 \notin E(\overline{G})$, then u_1x_i and $u_2x_j \in E(\overline{G})$, $i \neq j$. Hence, we get $G \cong G_1$.

Case 2 : $\varepsilon(\overline{A}) = \binom{n-2}{2} - 1$.

Then, $\overline{A} \cong K_{n-2}\backslash e$. Suppose $e = x_\ell x_m$, with x_ℓ and $x_m \in V(\overline{A})$. By Remark 3.1, we get $d(\overline{G}) = 3$ and hence any vertex of $\overline{H}$ must be joined to the same vertex x_ℓ or x_m , as otherwise $c(\overline{G})$ $> n-2$ or $d(\overline{G}) > 3$.

If $u_1u_2 \in E(\overline{H})$, then without loss of generality, we can take $u_1x_\ell \in E(\overline{G})$ and $u_2x_\ell \notin E(\overline{G})$, and so we get $G \cong G_3$.

If $u_1u_2 \notin E(\overline{H})$, again, without loss of generality, we can take u_1x_ℓ and $u_2x_\ell \in E(\overline{G})$ and hence we get $G \cong G_4$. This completes the proof of the theorem.

CONCLUSION AND SUGGESTION

The characterisation of the extremal graphs of $G \in$ G(n, 4, $n-2$) is as stated in the Theorem 3.6. We suggest the readers to investigate more general problem for $G \in G(n, c, \overline{c})$ and characterise the related extremal graphs.

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