EFFECTIVENESS OF EIGHTH ORDER RUNGE-KUTTA METHOD TO SOLVE THE MATHEMATICAL MODEL OF MALARIA DISEASE TRANSMISSION

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Abstrak. *Banyak permasalahan di lingkungan kehidupan kita yang dapat dibentuk ke dalam model matematika sehingga dapat dianalisis secara matematik. Salah satu permasalahan itu adalah kejadian endemi, seperti transmisi penyakit malaria. Model matematika transmisi penyakit malaria berbentuk system Persamaan Diferensial Biasa (PDB) non linier orde satu. Dalam tulisan ini akan dibahas efektivitas dan efisiensi metode Runge-Kutta orde delapan yang dibandingkan dengan metode Adams Bashforth-Moulton orde sembilan. Selain itu juga akan dicari sifat-sifat, formula, konvergenitas, dan format pemrograman MATLAB dari metode itu. Efektivitas suatu metode bergantung pada error. Sedangkan efisiensi bergantung pada waktu tempuh suatu metode untuk menyelesaikan masalah. Metode pengumpulan data yang digunakan adalah metode dokumentasi dan eksperimen. Hasil dari tulisan ini yaitu sifat dan formula metode Runge-Kutta orde delapan, pembuktian konvergensi metode tersebut secara teoritis, dan format pemrograman yang hasilnya digunakan untuk menentukan metode yang paling efektif dan efisien untuk menyelesaikan model transmisi penyakit malaria.*

Kata kunci : *Efektivitas, Efisiensi, Metode Runge-Kutta, Transmisi malaria.*

INTRODUCTION

Malaria remains a global health problem. In any given year, nearly ten percent of the global population will suffer a case of malaria [5]. If the disease is not treated immediately, human will be injury or even die. To study this situation, a mathematical model is needed to know the transmission of the disease. Such that we can choose the best preventive measures.

Mathematical model for malaria transmission which is developed by Chitnis [6], is a first order non linear system of ordinary differential equation (ODE). To solve the model we use numerical method. Runge-Kutta method is one of numerical method which tries to achieve a high degree of accuracy [9]. The order of Runge-Kutta method in this study is eight because it has not been researched yet. The previous researches of developed Runge-Kutta method are third order Runge-Kutta method [11], fourth order Runge-Kutta method [1], fifth order Runge-Kutta method

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[2], sixth order Runge-Kutta method [10], and seventh order Runge-Kutta method [4]. Therefore in this paper we will discuss the properties and formula of eighth order Runge-Kutta method, its convergence, its program in MATLAB, and its effectiveness and efficiency to solve malaria transmission by comparing to ninth Order Adams Bashforth-Moulton method [7].

Model is a simple description of reality that is expected to represent the current reality [8]. In mathematics, a model is formed in mathematic language, such as differential equation. For example a mathematical model for malaria transmission. The model which is developed by Chitnis [6] is as follows:

$$\frac{dS_h}{dt} = \Lambda_h + \psi_h N_h + \rho_h R_h - \lambda_h(t) S_h - f_h(N_h) S_h, \quad (1)$$
$$\frac{dE_h}{dt} = \lambda_h(t) S_h - \nu_h E_h - f_h(N_h) E_h, \quad (2)$$
$$\frac{dI_h}{dt} = \nu_h E_h - \gamma_h I_h - f_h(N_h) I_h - \delta_h I_h, \quad (3)$$
$$\frac{dR_h}{dt} = \gamma_h I_h - \rho_h R_h - f_h(N_h) R_h, \quad (4)$$
$$\frac{dS_v}{dt} = \psi_v N_v - \lambda_v(t) S_v - f_v(N_v) S_v, \quad (5)$$
$$\frac{dE_v}{dt} = \lambda_v(t) S_v - \nu_v E_v - f_v(N_v) E_v, \quad (6)$$
$$\frac{dI_v}{dt} = \nu_v E_v - f_v(N_v) I_v, \quad (7)$$

Where $f_h(N_h) = \mu_{1h} + \mu_{2h} N_h$ is the per capita density-dependent death and emigration rate for human and $f_v(N_v) = \mu_{1v} + \mu_{2v} N_v$ is the per capita density-dependent death for mosquitoes. Total human population size is N_h that is divided into sub population susceptible human (S_h), exposed human (E_h), infected human (I_h), and recovered human (R_h). While total vector population size is N_v that is divided into sub population S_v, E_v, dan I_v, with,

$$\frac{dN_h}{dt} = \Lambda_h + \psi_h N_h - f_h(N_h) N_h - \delta_h I_h,$$
$$\frac{dN_v}{dt} = \psi_v N_v - f_v(N_v) N_v,$$

and the force of infection from mosquitoes to humans and the force of infection from humans to mosquitoes respectively are λ_h and λ_v. With:

$$\lambda_h = b_h(N_h, N_v) \cdot \beta_{hv} \cdot \frac{I_v}{N_v} \quad dan \quad \lambda_v = b_v(N_h, N_v) \cdot (\beta_{vh} \cdot \frac{I_h}{N_h} + \tilde{\beta}_{vh} \cdot \frac{R_h}{N_h}) \tag{8}$$

b is the total number of mosquito bites on humans that is defined as: $b = b(N_h, N_v) = \frac{\sigma_v \sigma_h}{\sigma_v (N_v/N_h) + \sigma_h} N_v$. While b_h and b_v respectively are the number of bites per human per unit time and the number of bites per mosquito per unit time that is defined as: $b_h = b_h(N_h, N_v) = b(N_h, N_v)/N_h,\ \ and\ \ b_v = b_v(N_h, N_v) = b(N_h, N_v)/N_v$.

The state variables (Table 1) malaria model satisfy equations (1) up to (7). Parameter values for mathematical model of malaria disease transmission are written in [6]. In this study, the initial values of S_h, E_h, I_h, R_h, S_v, E_v, and I_v respectively are 400, 10, 30, 0, 1.000, 100, and 50.

Table 1: The state variable for the malaria model

Variable	Description	Unit
S_h	Number of susceptible human	human
E_h	Number of exposed human	human
I_h	Number of infected human	human
R_h	Number of recovered human	human
S_v	Number of susceptible vectors	mosquitoes
E_v	Number of exposed vectors	mosquitoes
I_v	Number of infectious vectors	mosquitoes

Here is a general useful definition in deriving the formula of eighth order Runge-Kutta method.

Definition 1. *Runge-Kutta method generally is defined by:*

$$y_{n+1} = y_n + h \sum_{i=1}^{m} b_i k_i \tag{9}$$

where,

$$k_i = f(x_n + c_i h, y_n + h\sum_{j}^{i-1} a_{ij}k_j), \quad i = 1, 2, \ldots, m \tag{10}$$

with assumption that:

$$c_i = \sum_{j=1}^{m} a_{ij}, dan \sum_{i=1}^{m} b_i = 1 \tag{11}$$

The value of a, b, and c can be written in Butcher array below.

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_m & a_{m1} & a_{m2} & \ldots & a_{mm-1} & \\ \hline & b_1 & b_2 & \ldots & b_{m-1} & b_m \end{array}$$

Now, in this study we will discuss the Runge-Kutta method which has eighth order ($m = 8$).

METHODS AND TECHNIQUES

Data collection methods of effectiveness of eighth order Runge-Kutta method for malaria transmission used documentation and experimental methods. In this study, method of documentation is used to find mathematical model for malaria transmission. While the experimental method is used to analyze output of the program in order to choose the most effective and efficient method both Runge-Kutta method and Adams Bashforth-Moulton method. The research techniques are as follow: (1) determining the properties of eighth order Runge-Kutta method; (2) deriving formula of eighth order Runge-Kutta method; (3) determining the convergence of the method; (4) using mathematical model for malaria transmission; (5) formulating model numerically; (6) making a pattern algorithm; (7) making program in MATLAB; (8) executing the program; (9) collecting the data such as error, iteration, time, and figure; (10) analyzing the data and comparing to Adams Bashforth-Moulton method; (11) concluding the most effective and efficient method.

THE RESULTS OF RESEARCH

Formula of eighth order Runge-Kutta methods has k_1, k_2, k_3, $\ldots, k_8$. We use equations (9), (10), and (11) to derive the formula of the method. After deriving the formula, we will have the properties of eighth order Runge-Kutta method such as Lemma 1. Furthermore, mathematicians have developed the properties of Runge-Kutta method. That is: $\sum_{i=1}^{m} b_i = 1 \ and \ c_i = \sum_{j=1}^{m} a_{ij}$.

Lemma 1. *Eighth Order Runge-Kutta method has the following properties:*

$$\sum_{i=2}^{8} b_i c_i^r = \frac{1}{r+1}, for \ r = 1, 2, 3, \ldots, 7 \tag{12}$$

$$\sum_{i=3}^{8} b_i (\sum_{j=2}^{i-1} c_j^r a_{ij}) = \frac{1}{(r+1)(r+2)}, for \ r = 1, 2, 3, 4, 5 \tag{13}$$

Proof. Based on Definition 1, the eighth order Runge-Kutta method is defined by $y_{n+1} = y_n + h \sum_{i=1}^{m} b_i k_i$ where $m = 8$,so we can write:

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5 + b_6k_6 + b_7k_7 + b_8k_8)$$

where,

$$\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + c_2h, y_n + ha_{21}k_1) \\
k_3 &= f(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2)) \\
k_4 &= f(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)) \\
k_5 &= f(x_n + c_5h, y_n + h(a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)) \\
k_6 &= f(x_n + c_6h, y_n + h(a_{61}k_1 + a_{62}k_2 + a_{63}k_3 + a_{64}k_4 + a_{65}k_5)) \\
k_7 &= f(x_n + c_7h, y_n + h(a_{71}k_1 + a_{72}k_2 + a_{73}k_3 + a_{74}k_4 + a_{75}k_5 + a_{76}k_6)) \\
k_8 &= f(x_n + c_8h, y_n + h(a_{81}k_1 + a_{82}k_2 + a_{83}k_3 + a_{84}k_4 + a_{85}k_5 + a_{86}k_6 + a_{87}k_7))
\end{aligned}$$

The value of $k_1, k_2, k_3, k_4, k_5, k_6, k_7$, and k_8 must be determined such that the previous equation will equal to Taylor algorithm. By expanding $y(x_{n+1})$ into x_n, we get,

$$\begin{aligned}
y(x_{n+1}) &= y(x_n) + hy^{(1)}(x_n) + \frac{1}{2!}h^2y^{(2)}(x_n) + \frac{1}{3!}h^3y^{(3)}(x_n) + \frac{1}{4!}h^4y^{(4)}(x_n) + \\
&\quad \frac{1}{5!}h^5y^{(5)}(x_n) + \frac{1}{6!}h^6y^{(6)}(x_n) + \frac{1}{7!}h^7y^{(7)}(x_n) + \frac{1}{8!}h^8y^{(8)}(x_n) + \\
&\quad \frac{1}{9!}h^9y^{(9)}(x_n) + \dots
\end{aligned}$$

After that, derivation of $y(x_n)$ is determined such that we have,

$$\begin{aligned}
y^{(1)}(x_n) &= f \\
y^{(2)}(x_n) &= f_x + f_y f \\
y^{(3)}(x_n) &= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(f_x + f_y f) \\
y^{(4)}(x_n) &= f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} + f_y(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + \\
&\quad \dots + ff_{yyy} \\
y^{(5)}(x_n) &= f_{xxxx} + 4ff_{xxxy} + 6f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy} + f_y(f_{xxx} + \\
&\quad 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}) + \dots + ff_{yyyy} \\
y^{(6)}(x_n) &= f_{xxxxx} + 5ff_{xxxxy} + 10f^2 f_{xxxyy} + 10f^3 f_{xxyyy} + 5f^4 f_{xyyyy} + \\
&\quad f^5 f_{yyyyy} + f_y(f_{xxxx} + 4ff_{xxxy} + 6f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy}) + \\
&\quad \dots + ff_{yyyyy} \\
y^{(7)}(x_n) &= f_{xxxxxx} + 6ff_{xxxxxy} + 15f^2 f_{xxxxyy} + 20f^3 f_{xxxyyy} + 15f^4 f_{xxyyyy} + \\
&\quad 6f^5 f_{xyyyyy} + f^6 f_{yyyyyy} + f_y(f_{xxxxx} + 5ff_{xxxxy} + 10f^2 f_{xxxyy} + \\
&\quad 10f^3 f_{xxyyy} + 5f^4 f_{xyyyy} + f^5 f_{yyyyy}) + \dots + ff_{yyyyyy} \\
y^{(8)}(x_n) &= f_{xxxxxxx} + 7ff_{xxxxxxy} + 21f^2 f_{xxxxxyy} + 35f^3 f_{xxxxyyy} + 35f^4 f_{xxxyyyy} + \\
&\quad 21f^5 f_{xxyyyyy} + 7f^6 f_{xyyyyyy} + f^7 f_{yyyyyyy} + f_y(f_{xxxxxx} + 6ff_{xxxxxy} + \\
&\quad 15f^2 f_{xxxxyy} + 20f^3 f_{xxxyyy} + 15f^4 f_{xxyyyy} + 6f^5 f_{xyyyyy}) + f^6 f_{yyyyyy} + \\
&\quad \dots + ff_{yyyyyyy}
\end{aligned}$$

Next, the derivation is shorten by defining the following notations,

$$\begin{aligned}
J &= f_x + f_y f \\
K &= f_{xx} + 2ff_{xy} + f^2 f_{yy} \\
L &= f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} \\
M &= f_{xxxx} + 4ff_{xxxy} + 6f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy} + \ldots \\
N &= f_{xxxxx} + 5ff_{xxxxy} + 10f^2 f_{xxxyy} + 10f^3 f_{xxyyy} + 5f^4 f_{xyyyy} + f^5 f_{yyyyy} + \ldots \\
O &= f_{xxxxxx} + 6ff_{xxxxxy} + 15f^2 f_{xxxxyy} + 20f^3 f_{xxxyyy} + 15f^4 f_{xxyyyy} + \\
&\quad 6f^5 f_{xyyyyy} + f^6 f_{yyyyyy} + \ldots \\
P &= f_{xxxxxxx} + 7ff_{xxxxxxy} + 21f^2 f_{xxxxxyy} + 35f^3 f_{xxxxyyy} + 35f^4 f_{xxxyyyy} + \\
&\quad 21f^5 f_{xxyyyyy} + 7f^6 f_{xyyyyyy} + f^7 f_{yyyyyyy} + \ldots
\end{aligned}$$

Therefore the expansion of $y(x_{n+1})$ can be written as,

$$\begin{aligned}
y(x_{n+1}) &= y(x_n) + hf + \frac{1}{2!}h^2 J + \frac{1}{3!}h^3(K + Jf_y) + \frac{1}{4!}h^4(L + Kf_y) + \\
&\quad \frac{1}{5!}h^5(M + Lf_y) + \frac{1}{6!}h^6(N + Mf_y) + \frac{1}{7!}h^7(O + Nf_y) + \\
&\quad \frac{1}{8!}h^8(P + Of_y) + \frac{1}{9!}h^9 y^{(9)}(x_n) + \ldots
\end{aligned}$$

By expanding k_1, k_2, k_3, $\ldots, k_8$ like expansion of Taylor series of two variables and substituting the notations J, K, L, M, N, O, and P, we get the new value of k_i. Then, we substitute equations of k_1, k_2, k_3,$\ldots, k_8$ to the equation of y_{n+1}. Next, by comparing the coefficients of both equations, we will get the properties of eighth order Runge-Kutta method. □

Corollary 1. *For step size h then the formula of eighth order Runge-Kutta method (RK8B1)is:*

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{120960}(5257k_1 + 25039k_2 + 9261k_3 + 20923k_4 + 20923k_5 + \\ &\quad 9261k_6 + 25039k_7 + 5257k_8) \end{aligned}$$

with,

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{h}{7}, y_n + \frac{1}{7}hk_1) \\ k_3 &= f(x_n + \frac{2}{7}h, y_n + \frac{h}{1323}(7538k_1 - 7160k_2)) \\ k_4 &= f(x_n + \frac{3}{7}h, y_n + \frac{h}{5978}(549k_1 + 4882k_2 - 2869k_3)) \\ k_5 &= f(x_n + \frac{4}{7}h, y_n + \frac{h}{427}(-693k_1 + 682k_2 - 211k_3 + 466k_4)) \\ k_6 &= f(x_n + \frac{5}{7}h, y_n + \frac{h}{378}(-79k_1 + 322k_2 + 224k_3 + 126k_4 - 323k_5)) \\ k_7 &= f(x_n + \frac{6}{7}h, y_n + \frac{h}{3577}(-2537k_1 + 2568k_2 + 1021k_3 + 511k_4 + \\ &\quad 511k_5 + 992k_6)) \\ k_8 &= f(x_n + h, y_n + \frac{h}{1502}(-61k_1 + 102k_2 + 428k_3 - 112k_4 + 126k_5 + \\ &\quad 242k_6 + 777k_7)) \end{aligned}$$

Proof. To prove the corollary, we can use Lemma 1. By solving the lemma we can determine the coefficients. Lemma 1 has 13 equations with 36 variables. The system can be solved by letting the value of c_1, c_2, c_3, $\ldots$, c_8, then substituting them into the properties of the method such that we get the value of b_1, b_2, b_3, $\ldots$, b_8 that satisfy $\sum_{i=1}^{8} b_i = 1$ and $c_i = \sum_{i=1}^{8} a_{ij}$. In this case we let $c_1 = 0$, $c_2 = \frac{1}{7}$, $c_3 = \frac{2}{7}$, $c_4 = \frac{3}{7}$, $c_5 = \frac{4}{7}$, $c_6 = \frac{5}{7}$, $c_7 = \frac{6}{7}$ and $c_8 = 1$. To find the value of $a_{32}, a_{42}, a_{43}, \ldots, a_{87}$, we must modify the equation (19) into $\sum_{i=2}^{7} c_j^r (\sum_{j=i+1}^{8} b_j a_{ji}) = \frac{1}{(r+1)(r+2)}$, for $r = 1, 2, 3, 4, 5$ and let $\sum_{j=i+1}^{8} b_j a_{ji} = A$ for $i = 2$, $\sum_{j=i+1}^{8} b_j a_{ji} = B$ for $i = 3$, $\ldots$, $\sum_{j=i+1}^{8} b_j a_{ji} = F$ for $i = 7$. Then, we have new equations in those variables. By solving the new equations we have the value of a_{21}, a_{31}, a_{32}, $\ldots$, a_{87}. It can be expressed as butcher array in table 3. $\square$

Table 2: Coefficient Matrix of RK8B1

0	0							
$\frac{1}{7}$	$\frac{1}{7}$	0						
$\frac{2}{7}$	$\frac{7538}{1323}$	$\frac{-7160}{1323}$	0					
$\frac{3}{7}$	$\frac{549}{5978}$	$\frac{4882}{5978}$	$\frac{-2869}{5978}$	0				
$\frac{4}{7}$	$\frac{-693}{427}$	$\frac{682}{427}$	$\frac{-211}{427}$	$\frac{466}{427}$	0			
$\frac{5}{7}$	$\frac{-79}{378}$	$\frac{322}{378}$	$\frac{224}{378}$	$\frac{126}{378}$	$\frac{-323}{378}$	0		
$\frac{6}{7}$	$\frac{-2537}{3577}$	$\frac{2568}{3577}$	$\frac{1021}{3577}$	$\frac{511}{3577}$	$\frac{511}{3577}$	$\frac{992}{3577}$	0	
1	$\frac{-61}{1502}$	$\frac{102}{1502}$	$\frac{428}{1502}$	$\frac{-112}{1502}$	$\frac{126}{1502}$	$\frac{242}{1502}$	$\frac{777}{1502}$	0
	$\frac{5257}{120960}$	$\frac{25039}{120960}$	$\frac{9261}{120960}$	$\frac{20923}{120960}$	$\frac{20923}{120960}$	$\frac{9261}{120960}$	$\frac{25039}{120960}$	$\frac{5257}{120960}$

Not only that, we can find the other formula by using the equal value of c but it has a minimum matrix as follows.

Corollary 2. *For step size h then the formula of eighth order Runge-Kutta method is:*

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{120960}(5257k_1 + 25039k_2 + 9261k_3 + 20923k_4 + \\ &\quad 20923k_5 + 9261k_6 + 25039k_7 + 5257k_8) \end{aligned}$$

dengan,

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{h}{7}, y_n + \frac{h}{7}k_1) \\ k_3 &= f(x_n + \frac{2}{7}h, y_n + \frac{h}{63}(-163k_1 + 181k_2)) \\ k_4 &= f(x_n + \frac{3}{7}h, y_n + \frac{h}{854}(621k_1 - 255k_3)) \end{aligned}$$

$$\begin{aligned}
k_5 &= f(x_n + \frac{4}{7}h, y_n + \frac{h}{427}(-350k_1 + 594k_4)) \\
k_6 &= f(x_n + \frac{5}{7}h, y_n + \frac{h}{126}(143k_1 - 53k_5)) \\
k_7 &= f(x_n + \frac{6}{7}h, y_n + \frac{h}{511}(279k_1 + 159k_6)) \\
k_8 &= f(x_n + h, y_n + \frac{h}{1502}(725k_1 + 777k_7))
\end{aligned}$$

Theorem 1. *Eighth order Runge-Kutta methods is a convergent method because it satisfies the following property* $||e_n|| \leq \frac{h^8 M_9}{362880\hat{L}}(e^{(x_n-x_0)\hat{L}} - 1)$, where $\hat{L}$ is Lipschitz constant.

Proof. The exact solution of a differential equation on $x = x_n$ is called $y(x_n)$ and numerical solution is called y_n. The numerical solution of Runge-Kutta method is got in the equation of y_{n+1}. Next, we will estimate how much the global error e_n.

Based on definition of global error [3], $e_n = y(x_n) - y_n$, so $e_{n+1} = y(x_{n+1}) - y_{n+1}$. We can write as:

$$\begin{aligned}
e_{n+1} &= e_n + h(y(x_n)^{(1)} - y_n^{(1)}) + \frac{1}{2!}h^2(y(x_n)^{(2)} - y_n^{(2)}) + \\
&\quad \frac{1}{3!}h^3(y(x_n)^{(3)} - y_n^{(3)}) + \frac{1}{4!}h^4(y(x_n)^{(4)} - \\
&\quad y_n^{(4)}) + \frac{1}{5!}h^5(y(x_n)^{(5)} - y_n^{(5)}) + \frac{1}{6!}h^6 \\
&\quad (y(x_n)^{(6)} - y_n^{(6)}) + \frac{1}{7!}h^7(y(x_n)^{(7)} - y_n^{(7)}) + \\
&\quad \frac{1}{8!}h^8(y(x_n)^{(8)} - y_n^{(8)}) + \frac{1}{9!}h^9 y(\eta)^{(9)}
\end{aligned}$$

By using Lipschitz requirement [3] and assumed $|y(\eta)^{(9)}| < M_9$ then,

$$\begin{aligned}
||e_{n+1}|| &\leq ||e_n + hL_1e_n + \frac{h^2}{2!}L_2e_n + \frac{h^3}{3!}L_3e_n + \\
&\quad \frac{h^4}{4!}L_4e_n + \frac{h^5}{5!}L_5e_n + \frac{h^6}{6!}L_6e_n + \\
&\quad \frac{h^7}{7!}L_7e_n + \frac{h^8}{8!}L_8e_n + \frac{h^9}{9!}M_9||
\end{aligned}$$

$$= \quad (1 + h\hat{L})||e_n|| + \frac{h^9}{362880}M_9$$

$\hat{L}$ is Lipschitz constant. Using the facts of $||e_n||$ and solving the inequality, we will have:

$$(1 + h\hat{L})^n \leq e^{nh\hat{L}} = e^{(x_n - x_0)\hat{L}}$$

so,

$$\begin{aligned} ||e_n|| &\leq \frac{h^8}{362880\hat{L}}M_9(e^{(x_n-x_0)\hat{L}} - 1) \\ \lim_{h\to 0} ||e_n|| &\leq \lim_{h\to 0} \frac{h^8}{362880\hat{L}}M_9(e^{(x_n-x_0)\hat{L}} - 1) \\ \lim_{h\to 0} ||e_n|| &\leq 0 \leftrightarrow \lim_{h\to 0} ||e_n|| = 0 \end{aligned}$$

where $\lim_{h\to 0} ||e_n|| \leq 0$ is always a positive numbers. Then, we can write that $\lim_{h\to 0} ||e_n|| = 0$. Therefore eighth order Runge-Kutta method is a convergent method. □

Test of Effectiveness of Eighth Order Runge-Kutta Method

Here is the results generated from the effectiveness program execution on MATLAB of Runge-Kutta and Adams Bashforth-Moulton methods.

Table 3: Data of Effectiveness RK8 dan ABM9

Iteration	Error in method		
	RK8B1	RK8B2	ABM9
100	0, 147911262777939	0, 147911263547030	**0,147910998122370**
735	**0,007174493565799**	0, 007174493752999	0, 007174502496213
1.000	**0,007898917337201**	0, 007898917404049	0, 007898918969971
1.650	0, 00654220253841	0, 00654220254819	**0,00654220248384**
5.000	0, 001487336759965	0, 001487336759965	**0,001487336742457**
10.000	0, 001153049869686	0, 001153049869686	**0,001153049869345**
100.000	0, 000622439069957	0, 000622439069957	**0,000622439069843**

Based on Table 4 we can analyze that Runge-Kutta method has bigger error than Adams Bashforth-Moulton for iterations 100, 1.650, 5.000, 10.000, and 100.000. Although for iteration 735 up to 1.000, Runge-Kutta method has less error. Thus, we can conclude that Eighth order Runge-Kutta method is not more effective method that Ninth order Adams Bashforth-Moulton method.

Test of Efficiency of Eighth Order Runge-Kutta Method

Here is the results generated from the efficiency program execution on MATLAB of Runge-Kutta and Adams Bashforth-Moulton method.

Table 4: Data of Efficiency RK8 dan ABM9

Output	Tolerance (e)	RK8B1	RK8B2	ABM9
Iteration	0,001	42.180	42.180	42.180
	0,0001	313.024	313.024	313.024
	0,00001	1.226.780	1.226.780	1.226.780
Time (second)	0,001	34,835	**25,599**	36,847
	0,0001	248,618	**240,932**	329,940
	0,00001	3294,257	**3275,148**	3518,133

Based on Table 5 we can analyze that Runge-Kutta and Adams Bashforth-Moulton method has equal value of iteration for all value of tolerance. Runge-Kutta method is faster to solve the model of malaria transmission for every tolerance. It is proven by the data on the table. So that, we can say Runge-Kutta method is more efficient that Adams Bashforth-Moulton method in solving the problem.

CONCLUSION

Based on the result of discussion, we can conclude that:

1. Eighth order Runge-Kutta method has the following properties,

$$\sum_{i=2}^{8} b_i c_i^r = \frac{1}{r+1}, for \ \ r = 1, 2, 3, \ldots, 7$$

$$\sum_{i=3}^{8} b_i (\sum_{j=2}^{i-1} c_j^r a_{ij}) = \frac{1}{(r+1)(r+2)}, for \ \ r = 1, 2, 3, 4, 5$$

2. One of the found formula of eighth order Runge-Kutta method is:

$$y_{n+1} = y_n + \frac{h}{120960}(5257k_1 + 25039k_2 + 9261k_3 + 20923k_4 + 20923k_5 + 9261k_6 + 25039k_7 + 5257k_8)$$

where,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{7}, y_n + \frac{1}{7}hk_1)$$

$$k_3 = f(x_n + \frac{2}{7}h, y_n + \frac{h}{1323}(7538k_1 - 7160k_2))$$

$$k_4 = f(x_n + \frac{3}{7}h, y_n + \frac{h}{5978}(549k_1 + 4882k_2 - 2869k_3))$$

$$k_5 = f(x_n + \frac{4}{7}h, y_n + \frac{h}{427}(-693k_1 + 682k_2 - 211k_3 + 466k_4))$$

$$k_6 = f(x_n + \frac{5}{7}h, y_n + \frac{h}{378}(-79k_1 + 322k_2 + 224k_3 + 126k_4 - 323k_5))$$

$$k_7 = f(x_n + \frac{6}{7}h, y_n + \frac{h}{3577}(-2537k_1 + 2568k_2 + 1021k_3 + 511k_4 + 511k_5 + 992k_6))$$

$$k_8 = f(x_n + h, y_n + \frac{h}{1502}(-61k_1 + 102k_2 + 428k_3 - 112k_4 + 126k_5 + 242k_6 + 777k_7))$$

3. Eighth order Runge-Kutta method is a convergent method.

4. Programming of eighth order Runge-Kutta method in solving the mathematical model of malaria disease transmission can be made and applied.

5. Eighth order Runge-Kutta method is not more effective than ninth order Adams Bashforth-Moulton method, but it is more efficient.

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